1 Discrete Time Systems

Consider a discrete-time system with $x[n]$ as input and $y[n]$ as output.

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

The following are some of the possible properties that a system can have:

Linearity

A **linear system** has the properties below:

1. **additivity**

$$x_1[n] + x_2[n] \longrightarrow \boxed{} \longrightarrow y_1[n] + y_2[n] \tag{1}$$

2. **scaling** (or **homogeneity**)

$$\alpha x[n] \longrightarrow \boxed{} \longrightarrow \alpha y[n] \tag{2}$$

Here, α is some constant.
Together, these two properties are known as **superposition**:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \boxed{} \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Time Invariance

A system is **time-invariant** if its behavior is fixed over time:

$$x[n - n_0] \longrightarrow \boxed{} \longrightarrow y[n - n_0] \tag{3}$$

Causality

A **causal** system has the property that $y[n_0]$ only depends on $x[n]$ for $n \in (-\infty, n_0]$. An intuitive way of interpreting this condition is that the system does not "look ahead."

Bounded-Input, Bounded-Output (BIBO) Stability

In a BIBO stable system, if $x[n]$ is bounded, then $y[n]$ is also bounded. A signal $x[n]$ is bounded if there exists an M such that $|x[n]| \leq M < \infty \; \forall n$.

2 Linear Time-Invariant (LTI) Systems

A system is LTI if it is both linear and time-invariant. We define the **impulse response** of an LTI system as the output $h[n]$ when the input $x[n] = \delta[n]$ where $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$.

An LTI system can be uniquely characterized by its impulse response $h[n]$. In addition, the following properties hold:

- An LTI system is causal iff $h[n] = 0 \; \forall n < 0$.
- An LTI system is BIBO stable iff its impulse reponse is absolutely summable:

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$$

Convolution Sum

Consider the following LTI system with impulse reponse $h[n]$:

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

Notice that we can write $x[n]$ as a sum of impulses:

$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

In addition, we know that:

$$\delta[n] \longrightarrow \boxed{} \longrightarrow h[n]$$

By applying the LTI property of our system, we get that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The expression $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is referred to as the **convolution sum** and can be written as $x[n] * h[n]$ or $(x * h)[n]$.

3

Determine if the following systems are linear, time-invariant, and/or causal.

a) $y[n] = 2x[-2+3n] + 2x[2+3n]$

Answer

linear, not time-invariant, not causal

- Linearity: Set the input to $\hat{x}[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. Then

$$\begin{aligned}
\hat{y}[n] &= 2\hat{x}[-2+3n] + 2\hat{x}[2+3n] \\
&= 2(\alpha_1 x_1[-2+3n] + \alpha_2 x_2[-2+3n]) + 2(\alpha_1 x_1[2+3n] + \alpha_2 x_2[2+3n]) \\
&= 2\alpha_1 x_1[-2+3n] + 2\alpha_1 x_1[2+3n] + 2\alpha_2 x_2[-2+3n] + 2\alpha_2 x_2[2+3n] \\
&= \alpha_1 \underbrace{(2x_1[-2+3n] + 2x_1[2+3n])}_{y_1[n]} + \alpha_2 \underbrace{(2x_2[-2+3n] + 2x_2[2+3n])}_{y_2[n]} \\
&= \alpha_1 y_1[n] + \alpha_2 y_2[n]
\end{aligned}$$

- Time invariance:
  Let $\hat{x}[n] = x[n-n_0]$ be a delayed input signal. Then, the corresponding output $\hat{y}[n]$ is equal to $2x[-2+3n-n_0] + 2x[2+3n-n_0]$
  However, we can see that $\hat{y}[n] \neq y[n-n_0] = 2x[-2+3(n-n_0)] + 2x[2+3(n-n_0)]$
- Causality: Note that $y[0] = 2x[-2] + 2x[2]$ depends on $x[2]$, so the system is not causal.

b) $y[n] = 4^{x[n]}$

Answer

non-linear, time-invariant, causal

- Linearity: Let $\hat{x}[n] = 2x[n]$. Then $\hat{y}[n] = 16^{x[n]} \neq 2y[n]$. Thus, the system is not linear

- Time invariance: Let $\hat{x}[n] = x[n-n_0]$. Then $\hat{y}[n] = 4^{\hat{x}[n]} = 4^{x[n-n_0]} = y[n-n_0]$, so the system is time-invariant.

- Causality: Note that $y[n_0]$ depends on $x[n_0]$ only, and not on any $x[n]$ with $t > n_0$. The system is therefore causal.

Additional practice:

c) $y[n] - y[n-1] = x[n] - x[n-1] - x[n-2]$

Answer

linear, time-invariant, causal

- Linearity: Let $x_1[n]$ and $x_2[n]$ be inputs with corresponding outputs $y_1[n]$ and $y_2[n]$ Set the input to $\hat{x}[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. Then

$$\begin{aligned}\hat{y}[n] - \hat{y}[n-1] &= \hat{x}[n] - \hat{x}[n-1] - \hat{x}[n-2] \\ &= (\alpha_1 x_1[n] + \alpha_2 x_2[n]) - (\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]) - (\alpha_1 x_1[n-2] + \alpha_2 x_2[n-2]) \\ &= \alpha_1(x_1[n] - x_1[n-1] - x_1[n-2]) + \alpha_2(x_2[n] - x_2[n-1] - x_2[n-2]) \\ &= \alpha_1(y_1[n] - y_1[n-1])_+ \alpha_2(y_2[n] - y_2[n-1])\end{aligned}$$

  Note that this is true for ar all n.

- Time invariance: Consider input $x[n]$ and corresponding output $y[n]$ Let $\hat{x}[n] = x[n-n_0]$. The corresponding output $\hat{y}[n]$ follows

$$\begin{aligned}\hat{y}[n] - \hat{y}[n-1] &= \hat{x}[n] - \hat{x}[n-1] - \hat{x}[n-2] \\ &= x[n-n_0] - x[n-n_0-1] - x[n-n_0-2] \\ &= y[n-n_0] - y[n-n_0-1].\end{aligned}$$

  Therefore, the system is time-invariant.

- Causality: Note that $y[n_0] - y[n_0-1]$ depends only on $x[n_0]$, $x[n_0-1]$, $x[n_0-2]$, and $n_0, n_0-1, n_0-2 \leq n_0$ so the system is causal (no output depends on a future input).

d) $y[n] = x[n] + nx[n-1]$

Answer

linear, not time-invariant, causal

- Linearity:Let $x_1[n]$ and $x_2[n]$ be inputs with corresponding outputs $y_1[n]$ and $y_2[n]$ Set the input to $\hat{x}[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$. We check the system is linear:

$$\begin{aligned}\hat{y}[n] &= \hat{x}[n] + n\hat{x}[n-1] \\ &= \alpha_1 x_1[n] + \alpha_2 x_2[n] + n(\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]) \\ &= \alpha_1(x_1[n] + nx_1[n-1]) + \alpha_2(x_2[n] + nx_2[n-1]) \\ &= \alpha_1 y_1[n] + \alpha_2 y_2[n].\end{aligned}$$

- Time invariance: Let $x[n]$ be an input with output $y[n]$. Set $\hat{x}[n] = x[n-n_0]$ for $n_0 \neq 0$ and note that $y[n-n-0] = x[n-n_0] + (n-n_0)x[n-n_0-1]$ but $\hat{y}[n] = \hat{x}[n] + n\hat{x}[n-1] = x[n-n_0] + nx[n-n_0-1]$. Therefore $\hat{y}[n] \neq y[nn_0]$, and the system is not time invariant.
- Causality: Observe that $y[n_0]$ only depends on x at n_0 and $n_0 - 1$, so it does not depend on any future x. The system is causal. a

e) $y[n] = 2^n \cos(x[n])$

Answer

not linear, not time-invariant, causal

- Linearity: Suppose $x[n]$ is an input with corresponding output $y[n]$, and let $\hat{x}[n] = 2x[n]$. Then

$$\begin{aligned}\hat{y}[n] &= 2^n \cos\big(\hat{x}[n]\big) \\ &= 2^n \cos\big(2x[n]\big) \\ &\neq 2(2^n)\cos x[n].\end{aligned}$$

  Therefore, the system is not linear.

- Time invariance: Consider the sequence $\hat{x}[n] - x[n-n_0]$ for some $n_0 \neq 0$. Note that

$$\begin{aligned}\hat{y}[n] &= 2^n \cos\big(\hat{x}[n]\big) \\ &= 2^n \cos\big(x[n-n_0]\big) \\ &\neq 2^{n-n_0} \cos\big(x[n-n_0]\big) = y[n-n_0].\end{aligned}$$

  This shows the system is not time invariant.

- Causality: Observe that $y[n_0]$ does not depend on any inputs $x[n]$ for $n > n_0$, so the system is causal.

4 Convoluted Convolution

a) Show that convolution is commutative. That is, show that $(x * h)[n] = (h * x)[n]$.

Answer

$$\begin{aligned}
(x * h)[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{m=-\infty}^{\infty} x[n-m]h[m] && \text{Let } m = n-k. \\
&= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= (h * x)[n]
\end{aligned}$$

b) Show that $\delta[n]$ is a convolution identity. That is, show that $(x * \delta)[n] = x[n]$.

Answer

Since convolution is commutative, we know that $(x * \delta)[n] = (\delta * x)[n]$.

$$(\delta * x)[n] = \sum_{k=-\infty}^{\infty} \delta[k]x[n-k]$$

Since $\delta[k] = 0$ for all $k \neq 0$, it follows that

$$(\delta * x)[n] = \delta[0]x[n] = x[n]$$

Additional Practice:

c) Show that convolution by $\delta[n-n_0]$ shifts $x[n]$ by n_0 steps to the right.

Answer

Since convolution is commutative $x[n] * \delta[n-n_0] = \delta[n-n_0] * x[n]$.

$$\delta[n-n_0] * x[n] = \sum_{k=-\infty}^{\infty} \delta[k-n_0]x[n-k]$$

Then since $\delta[k-n_0] = 0$ for all $k \neq n_0$, it follows that

$$\delta[n-n_0] * x[n] = \delta[0]x[n-n_0] = x[n-n_0]$$

d) Show that convolution is distributive. In other words, show that $(x * (h_1 + h_2))[n] = (x * h_1)[n] + (x * h_2)[n]$.

Answer

Since multiplication is distributive, it follows that convolution is distributive

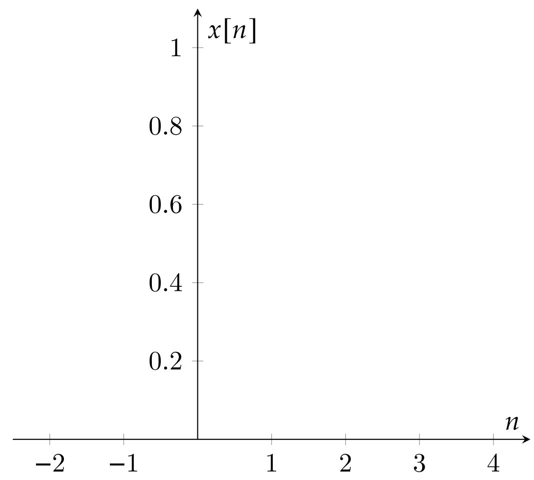
$$\begin{aligned}(x * (h_1 + h_2))[n] &= \sum_{k=-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k] \\ &= (x * h_1)[n] + (x * h_2)[n]\end{aligned}$$

5 Mystery System

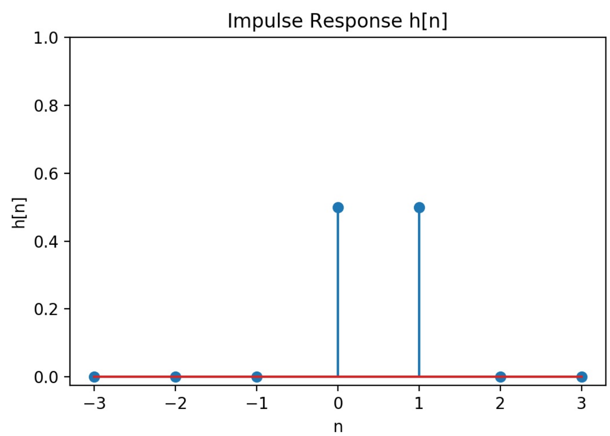
Consider an LTI system with the following impulse response:

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

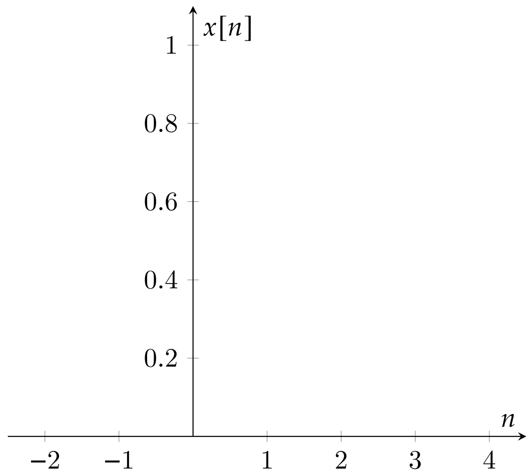
a) Create a sketch of this impulse reponse.



Answer



b) What is the output of our system if the input is the unit step $u[n]$?



Answer

$$y[n] = (u * h)[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k] = \sum_{k=0}^{\infty} h[n-k]$$

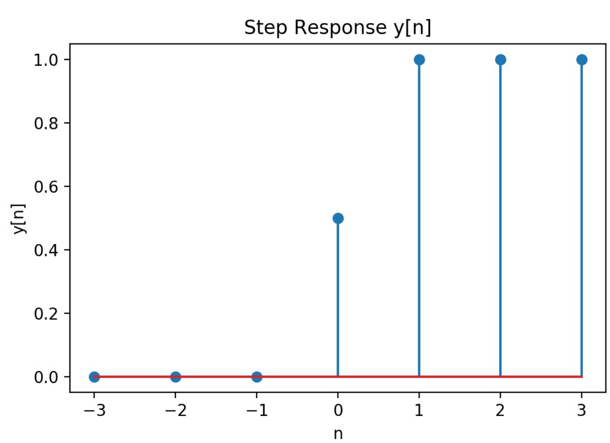
For $n < 0$, $y[n] = 0$. When $n > 0$,

$$y[0] = \sum_{k=0}^{\infty} h[-k] = h[0] = 0.5$$

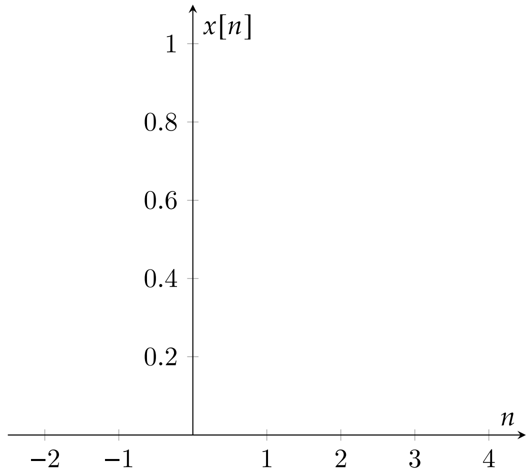
$$y[1] = \sum_{k=0}^{\infty} h[1-k] = h[0] + h[1] = 1$$

$$y[n] = \sum_{k=0}^{\infty} h[n-k] = h[0] + h[1] + \ldots + h[n] = 1 \text{ for } n > 1.$$

The output $y[n]$ is shown below.



c) What is the output of our system if our input is $x[n] = (-1)^n u[n]$?

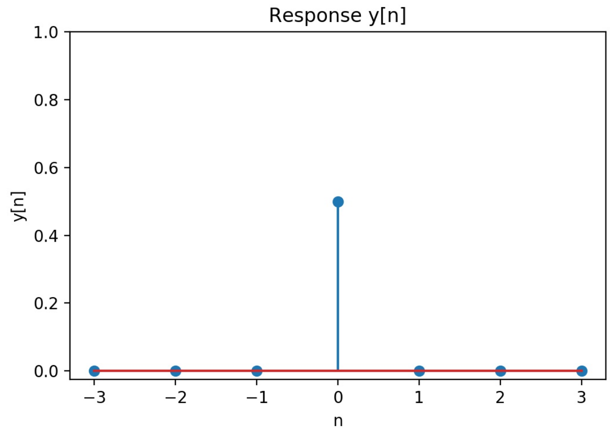


Answer

$$y[n] = (u * h)[n] = \sum_{k=-\infty}^{\infty} x[n]h[n-k] = \sum_{k=0}^{\infty}(-1)^k h[n-k]$$

For $n < 0$, $y[n] = 0$. When $n > 0$,

$$\begin{aligned}
y[0] &= \sum_{k=0}^{\infty} h[-k] = h[0] = 0.5 \\
y[1] &= \sum_{k=0}^{\infty} h[1-k] = h[0] - h[1] = 0 \\
y[2] &= \sum_{k=0}^{\infty} h[2-k] = h[2] - h[1] + h[0] = 0 \\
&\vdots \\
y[n] &= 0 \text{ for } n > 0.
\end{aligned}$$



d) This system is called the two-point simple moving average (SMA) filter. Based on the previous parts, why do you think it bears this name?

Answer

The output of the system at each timestep n is the average of $x[n]$ and $x[n-1]$. To show this formally, we can look at the convolution $y = x * h$

$$\begin{aligned} y[n] = (x * h)[n] &= x * (\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]) \\ &= \frac{1}{2}x[n] + \frac{1}{2}x[n-1] \end{aligned}$$

This sort of system can be used in areas like technical analysis to gain insight into stock prices and trends (usually these methods would use a longer window than just two days). There are also other variants used like the exponential moving average (EMA) filter.