EECS 16A Designing Information Devices and Systems I
Summer 2020 Midterm 1

1. Pledge of Academic Integrity (2 points)
By my honor, I affirm that:

(1) this document, which I will produce for the evaluation of my performance, will reflect my original, bona fide work;

(2) as a member of the UC Berkeley community, I have acted and will act with honesty, integrity, and respect for others;

(3) I have not violated—nor aided or abetted anyone else to violate—nor will I—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) I have not committed, nor will I commit, any act that violates—nor aided or abetted anyone else to violate—the UC Berkeley Code of Student Conduct.

Write your name and the current date as an acknowledgement of the above. (See Gradescope)

2. Administrivia (2 points)
I know that I will lose 2^n points for every n minutes I submit after the exam submission grace period is over.

For example, if the exam becomes available at my personalized link at 7:10 p.m. PT; the grace period will expire at 9:15 p.m. PT. If my submission is timestamped at 9:16 p.m. PT, I will lose 2 points; if it is timestamped at 9:18 p.m. PT, I will lose 8 points.

○ Yes

3. What is one of your hobbies? (2 points)

4. Tell us about something you're proud of this summer. (2 points)

5. Eeveelution (20 points)

(a) With your newly acquired knowledge of linear algebra and system modelling, you've got the opportunity to work in the legendary Professor Oak's Lab. His latest research delves into the evolution of Pokémon.

Pokémon evolution is studied using hit points (HP) and combat power (CP). Thus, each Pokémon can be represented with a vector of the form $\begin{bmatrix} h \\ c \end{bmatrix}$ where h stands for HP and c stands for CP.

Our study considers Eevee's evolution into Vaporeon, Jolteon, and Flareon, which can be represented by the following transformations:

$$V\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 15c + 15h \\ 4c + 6h + 16 \end{bmatrix}$$

$$J\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 15c \cdot h \\ 4c^2 + 10h \end{bmatrix}$$

$$F\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 14c + 8h \\ 7c + 16h \end{bmatrix}$$

Which of the above transformations are linear in $\begin{bmatrix} h \\ c \end{bmatrix}$? Select all that apply.

(1) F

(2) V

(3) J

Answer: (1)

(b) Professor Oak's prototype 'Eevolver' can only accept transformations in the form of matrices. Eevee can evolve into the rare psychic Pokémon Espeon through the following transformation:

$$E\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 2h \\ -5c + 29h \end{bmatrix}$$

The Professor has tasked you to come up with a matrix P such that $E\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = P\begin{bmatrix} h \\ c \end{bmatrix}$

Find a correct representation of the P matrix.

Answer: $\begin{bmatrix} 2 & 0 \\ 29 & -5 \end{bmatrix}$

(c) Professor Oak programs three new transformation matrices into his 'Eevolver':

- *Umbreon Transformation*, represented by the 2×2 matrix U
- *Leafeon Transformation*, represented by the 2×2 matrix L
- *Glaceon Transformation*, represented by the 2×2 matrix G

Starting with a Pokémon represented by $\vec{p} \in \mathbb{R}^2$, the Professor first applies the **Umbreon Transformation**. Then he applies the **Leafeon Transformation**, and last, he applies the **Glaceon Transformation**. What expression, in terms of L, G, U, and $\vec{p}$, represents the Pokémon after this series of transformations?

Answer: $GLU\vec{p}$

Professor Oak wants to try repeatedly using his 'Eevolver' on the same Pokémon, but he isn't sure which transformation matrix to try. To help him decide, he asks to you find the eigenvalues and associated eigenspaces of the following matrix:

$$T = \begin{bmatrix} 16 & 6 \\ -30 & -11 \end{bmatrix}$$

(d) State the equation you would solve to find the eigenvalues.

Answer: $\lambda^2 - 5\lambda + 4 = 0$

(e) Professor Oak tells you that one of the eigenvalues is 1. Find the associated eigenspace.

Answer: $\text{span}\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$

6. Snackable Linear Combinations (12 points)

(a) You want to make some snacks at home, with the ingredients you have. You limit your options to cookies, bread, muffins and donuts. The table below shows the ingredients needed for each snack.

Cookies		**Bread**		**Muffins**		**Donuts**	
INGREDIENTS		INGREDIENTS		INGREDIENTS		INGREDIENTS	
Eggs	2	Eggs	2	Eggs	2	Eggs	3
Flour (lb)	4	Flour (lb)	2	Flour (lb)	3	Flour (lb)	3
Butter (tbsp)	3	Butter (tbsp)	3	Butter (tbsp)	5	Butter (tbsp)	2
Sugar (tbsp)	2	Sugar (tbsp)	4	Sugar (tbsp)	5	Sugar (tbsp)	5

You have 21 eggs, 12 lb flour, 34 tbsps butter, 23 tbsps sugar available. You want to use all of these ingredients completely. Frame this problem in the form $A\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} x_c \\ x_b \\ x_m \\ x_d \end{bmatrix}$, and x_c, x_b, x_m, x_d represent the quantity of cookies, bread, muffins, and donuts, respectively. Choose the A, $\vec{b}$ pair that can be used to solve for the correct quantities of each snack.

Answer: $\begin{bmatrix} 2 & 2 & 2 & 3 \\ 4 & 2 & 3 & 3 \\ 3 & 3 & 5 & 2 \\ 2 & 4 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_c \\ x_b \\ x_m \\ x_d \end{bmatrix} = \begin{bmatrix} 21 \\ 12 \\ 34 \\ 23 \end{bmatrix}$

(b) You are given a new A and $\vec{b}$. Shown below are 3 steps of Gaussian Elimination performed on the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 0 & 2 & 5 & 3 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 & 6 \end{array}\right] \overrightarrow{??} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 5 & -3 \end{array}\right]$$

Find the correct row reduction steps.

Answer: (i) $R_2 \leftrightarrow R_3$ (ii) $R_3/2$ (iii) $R_4 = 2R_3 - R_4$

(c) You later find a new recipe book that you like better. The new A matrix created from these recipes is shown below. Find the basis vector for the nullspace of this matrix. Note: You can assume that a and b are nonzero.

$$A = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 2 & 0 \\ 0 & 0 & b & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer: $\left\{ \begin{bmatrix} -a \\ \frac{12}{b} \\ \frac{-6}{b} \\ 1 \end{bmatrix} \right\}$

7. Code Breaker (12 points)

You've been pranked! Your friends have stolen your things and have hidden them all over town. They are communicating messages back and forth that contain coordinates representing the locations of your missing stuff. Unfortunately, your friends are pretty smart, so they've used an encoding matrix to encode the vector of coordinates,

$$\vec{y} = A\vec{x},$$

where $\vec{y}$ is the encoded message, $\vec{x}$ is the original vector of coordinates, and A is the encoding matrix. But not to worry! Your 16A knowledge should help you break the code and find your stuff.

(a) You've successfully intercepted both the original and encoded versions of one of the locations. If the original vector, $\vec{x}$, belongs to $\mathbb{R}^2$ and the encoded vector, $\vec{y}$, belongs to $\mathbb{R}^6$, what are the dimensions of the encoding matrix, A?

Answer: 6×2

(b) What is the minimum number of pairs of original and encoded messages that you'd need to intercept in order to determine the encoding matrix?

Answer: 2

(c) We have successfully intercepted n pairs of messages, where n is greater than or equal to the minimum number of pairs needed to determine the encoding matrix. Which of the following must be true about the n intercepted $\vec{x}$ vectors (original messages) if we want to recover the encoding matrix? Select all that apply.

(1) They must all be linearly independent
(2) They must span $\mathbb{R}^2$
(3) They must form a basis for $\mathbb{R}^2$
(4) None of these
(5) The $2 \times n$ matrix containing these vectors as columns must have rank 2

Answer: (2), (5)

8. Fundamental Subspaces (12 points)

(a) You have a 3×3 matrix $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$. After performing some row operations, the resulting matrix is:

$$\begin{bmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

Non-zero entries are indicated by $*$. Construct a basis for the columnspace of the matrix A using $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

Answer: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

(b) What are the rank and nullity of the given matrix B where $*$ are non-zero values?

$$B = \begin{bmatrix} 0 & 0 & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer: $\text{Rank}(B) = 2, \text{Nullity}(B) = 4$

(c) Let's turn our attention to a more general case. Let M be a matrix with m rows and n columns. You visualize the nullspace of this matrix using computer software and see a plane in a three-dimensional space. You then try to visualize the columnspace of the matrix, but cannot as the software reports that it is unable to plot 9-dimensional vectors. Determine m, n, and the rank of M.

Answer: $m = 9$, $n = 3$, $\text{Rank}(M) = 1$

9. Pumps Problem (16 points)

(a) For the transition diagram below, which values of a, b, c would lead to a conservative system? If this is not possible, please state why.

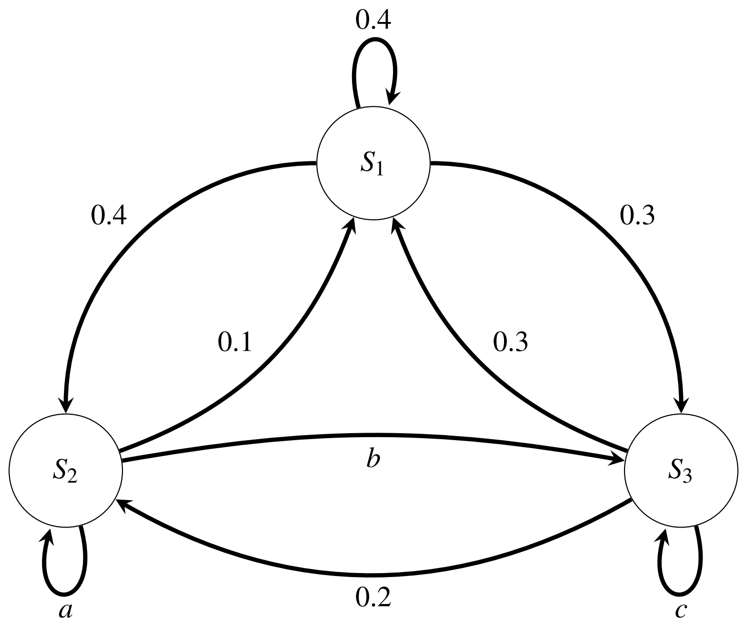

Figure 1: Transition Diagram

Answer: Not possible.

(b) For a different system given by transition matrix:

$$T = \begin{bmatrix} 0.5 & 4.0 & 1.75 \\ 2.0 & 4.0 & -7.0 \\ 1.0 & 2.0 & -2.5 \end{bmatrix}$$

and state vector $\vec{x}[n] \in \mathbb{R}^3$ at time n ≥ 1 given by

$$\vec{x}[n] = T\vec{x}[n-1]$$

What is the vector $\vec{x}$ such that the current state equals the previous state? In other words what is the $\vec{x}$ such that $\vec{x} = \vec{x}[n] = \vec{x}[n-1]$?

Answer: $\alpha \left\{ \begin{bmatrix} 3.5 \\ 0 \\ 1 \end{bmatrix} \right\}, \alpha \in \mathbb{R}^3$

(c) For a general system given by a transition matrix $B \in \mathbb{R}^{n\times n}$, which statements, if true, must imply that there exists a non-zero $\vec{x} \in \mathbb{R}^n$ such that $\vec{x} = B\vec{x}$? Choose all that apply.

(1) The columns of B are linearly dependent.

(2) $B - I$ has a non-trival nullspace.

(3) $B - I$ has a trivial nullspace.

(4) The columns of $B - I$ are linearly dependent.

(5) There is an eigenvalue $\lambda = 1$.

Answer: (2), (4), (5)

(d) Consider a state transition matrix $A \in \mathbb{R}^{3\times 3}$ with eigenvalue/eigenvector pairings:

$$\lambda_1 = 0.75, \vec{v}_1 \quad \lambda_2 = 1, \vec{v}_2 \quad \lambda_3 = 5, \vec{v}_3$$

where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent. We can express any initial state vector $\vec{x}[0] \in \mathbb{R}^3$ as a linear combination of these vectors. In other words:

$$\vec{x}[0] = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \vec{\alpha}$$

For what values of $\vec{\alpha}$ will this system converge to a non-zero steady state? Choose all that apply.

(1) $\vec{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$ (2) $\vec{\alpha} = \begin{bmatrix} 2 \\ 0.5 \\ 0.5 \end{bmatrix}$ (3) $\vec{\alpha} = \begin{bmatrix} -8 \\ 4 \\ 0 \end{bmatrix}$ (4) $\vec{\alpha} = \begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix}$ (5) $\vec{\alpha} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

Answer: (3)

10. Simple Functions of a Matrix (12 points)

Characterizing the eigenvalues and eigenvectors of a matrix A starts with solving the characteristic equation:

$$\det(A - \lambda I) = 0$$

This expression tells us a great deal about the structure embedded in A, which we will explore below. Let

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$

(a) Compute the characteristic equation, simplifying it to a polynomial function of λ.

Answer: $\lambda^2 - 5\lambda - 5 = 0$

In 1853, mathematicians Arthur Cayley and William Hamilton created a theorem centered around the characteristic equation:

Theorem 0.1: [Cayley-Hamilton] Every square matrix satisfies its own characteristic equation.

We state this theorem without proof. As you advance in your studies of linear algebra and linear systems, you will encounter this theorem again, as well as the tools necessary to prove its correctness, but for now just take this as a fact. For this section, we give you a new matrix B:

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

The characteristic equation for this matrix is $\lambda^2 - 5\lambda + 5 = 0$. What does it mean that a matrix should satisfy its own characteristic equation? The equation above is a function of λ, but what if we made it a function of B?

(b) What does the Cayley-Hamilton Theorem imply about the expression $B^2 - 5B + 5I$? Select all that apply.

(1) It equals λ
(2) It is composed of the eigenvectors of B
(3) It is undefined
(4) It equals B
(5) It equals $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Answer: (5)

(c) Use the Cayley-Hamilton Theorem to compute $2B^3 - 10B^2 + 10B - 2I$.

Answer: $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

11. A (Solar) System Far, Far Away (20 points)

As a wandering bounty hunter, you have recently purchased a spaceship, Swordfish I, initially positioned at (x_0, y_0, z_0), which can be moved to some final position (x_1, y_1, z_1), where the motion is characterized by matrix A.

(a) Assume that (x_0, y_0, z_0) and (x_1, y_1, z_1) are related by

$$A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 5 & 10 \\ -4 & -8 \\ 13 & 26 \end{bmatrix}.$$

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ does not depend on z_0. Choose **all** the options that are true.

(1) The rows of A are linearly independent.
(2) The columnspace of A is a subspace of $\mathbb{R}^3$.
(3) The columns of A are linearly independent.
(4) The rank of A is 2.
(5) The columns of A span the entire $\mathbb{R}^3$.

Answer: (2)

(b) You want to be able to navigate to any final position (x_1, y_1, z_1) by choosing your initial position (x_0, y_0, z_0). You start with the following equations representing your spaceship:

$$A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 6 & -6 \\ -5 & 0 \\ 15 & -15 \end{bmatrix}.$$

Consider the set of all possible final positions that can be reached from the set of all initial positions. What geometric object does this set form?

Answer: All possible values of (x_1, y_1, z_1) form a 2D plane.

(c) You notice that the final position (x_1, y_1, z_1) does not depend on the initial z-position, z_0. You upgrade your ship, modifying the matrix A by adding a third column. Your upgraded ship obeys the following equations:

$$A \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 0 & 3 \\ 4 & 5 & 0 \end{bmatrix}$$

Choose **all** the options that are true.

(1) The matrix A is non-invertible.
(2) The ship can reach any final position in the $\mathbb{R}^3$ space, if the right initial position (x_0, y_0, z_0) is used.
(3) The columns of A form a basis for $\mathbb{R}^3$.
(4) No two unique initial positions (x_0, y_0, z_0) result in the same final position (x_1, y_1, z_1).
(5) The dimension of the geometric object formed by the set of all possible values of (x_1, y_1, z_1) exceeds rank of A.

Answer: (2), (3), (4)

(d) Your upgraded ship opens a lot of doors for you; you decide to go explore the ($\mathbb{R}^3$) universe. You don't want to miss a single sight, and hope that all points in the universe will be reachable by your ship. As before, you know the current position of your ship can be described by a 3-dimensional vector $\vec{s} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Your ship's navigation system can be represented by the following model which relates the position to a control input vector $\vec{u}$:

$$\vec{s} = B\vec{u},$$

You decide to modify the the matrix B from its default settings. You would like to choose B such that you can reach any possible final position $\vec{s}$ within $\mathbb{R}^3$ by controlling the input $\vec{u}$, which is not necessarily in $\mathbb{R}^3$. Which B, $\vec{u}$ pairs will allow you to explore the whole universe? Choose **all** the options that work.

(1) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & 4 & 5 \\ 1 & 0 & 4 & 1 \\ 5 & 3 & 0 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$

(2) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & 4 \\ 1 & 0 & 4 \\ 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

(3) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & -5 \\ 1 & 0 & 1 \\ 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

(4) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & 5 & -5 \\ 1 & 0 & 1 & 1 \\ 5 & 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$

Answer: (1), (2)

(e) You decide to upgrade to a new ship, the Swordfish II, a ship with a system that can be expressed by the following model:

$$\vec{s}[t+1] = \vec{s}[t] + \vec{b}u[t],$$

where $\vec{s}[t]$ is the current position of the ship at time t, $\vec{s}[t+1]$ is the next position of the ship (at time $t+1$), vector $\vec{b}$ helps define the motion of the ship, and $u[t]$ represents a user-controlled scalar input at time t. As you fly, you notice another ship, the Hammerhead, with the same flight system getting closer on your navigation display – at time $t = 0$ it is at $(x,y,z) = (-4,2,4)$, that is, $\vec{s}_{\text{other ship}}[0] = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$.

You communicate with the captain of the other ship, and realize that:

$$\vec{b}_{\text{your ship}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_{\text{your ship}}[t] = -2$$

$$\vec{b}_{\text{other ship}} = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}, u_{\text{other ship}}[t] = 2$$

Both ships are on autopilot, so $u_{\text{your ship}}$ and $u_{\text{other ship}}$ will not change over time. For which of the following starting coordinates for your ship, $\vec{s}_{\text{your ship}}[0]$, if any, is it possible that your ship and the other will collide at some time in the future ($t > 0$)? Select all that apply.

(1) $(-8,-6,5)$

(2) $(0,10,4)$

(3) $(-12,-14,4)$

(4) $(-8,-6,4)$

(5) $(4,18,4)$

Answer: (3), (4)

12. Image Transformation (12 points)

Laura, an aspiring photographer, took a picture of the Golden Gate Bridge last week. However, it didn't turn out as perfect as she wanted it. So, she started using Adobe Photoshop to edit the image. Unfortunately, she left her computer alone for a few minutes and her dog accidentally sat on the keyboard! The image now looks like this.

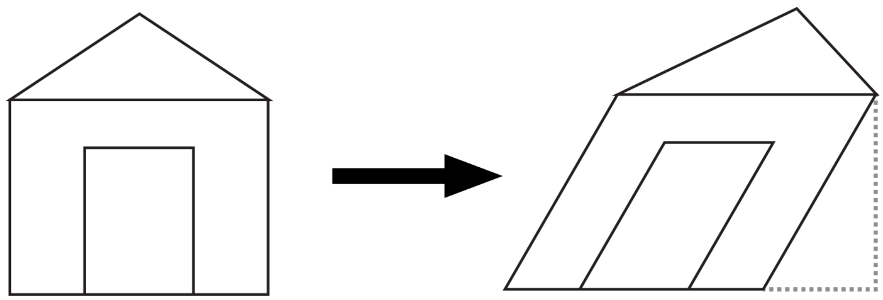

Figure 2: Image Transformation

No matter what she does, Laura can't seem to get the original image back. However, she figures out that her dog applied a transformation, T, to the original image of this form:

$$\begin{bmatrix} x_{new} \\ y_{new} \\ 1 \end{bmatrix} = T \begin{bmatrix} x_{orig} \\ y_{orig} \\ 1 \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{orig} \\ y_{orig} \\ 1 \end{bmatrix}$$

(a) Laura asks her friend for help and together they determine 3 out of the 6 unknowns of the the transformation matrix that was applied:

$$T = \begin{bmatrix} 3 & t_1 & t_2 \\ t_3 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

Laura measures the following pairs of pixel locations:

(x_{orig}, y_{orig})	(x_{new}, y_{new})
$(1,1)$	$(14,9)$
$(2,1)$	$(17,11)$
$(5,2)$	$(32,18)$
$(3,4)$	$(38,16)$

Use this information to solve for the remaining unknowns in the matrix T.

Answer: $t_1 = 6, t_2 = 5, t_3 = 2$

(b) Now that Laura knows the transformation her dog applied, she decides to try to undo the transformation using its inverse. However, she's not sure if the inverse is unique.

Prove that if R and X are both inverses of T, then $R = X$. Assume that R, X, and T are all $n \times n$ matrices. Finish the proof by filling in the blanks from the options below. Remember, each equation should follow directly from the previous equation in the proof. Options can be used more than once.

$$XT = I$$
$$\boxed{?}\,\boxed{?}\,\boxed{?} = \boxed{?}\,\boxed{?}$$
$$XI = R$$
$$X = R$$

(1) T (2) R (3) X (4) I

Answer: $XTR = IR$

(c) Laura sets up an augmented matrix of the form

$$\left[\begin{array}{c|c} T & X \end{array}\right]$$

where T, R, and X are all 4×4 matrices. After applying Gaussian Elimination, Laura gets the following augmented matrix:

$$\left[\begin{array}{c|c} I & R \end{array}\right]$$

What equation did she show to be true?

Answer: $TR = X$

13. Blurry Images (16 points)

Let $\vec{x} \in \mathbb{R}^n$ be a sharp image. We use a camera to take a picture of $\vec{x}$, but our camera is out of focus so our captured image, $\vec{y} \in \mathbb{R}^n$, is blurry. We can represent the blurring function as a linear transformation, B:

$$\vec{y} = B\vec{x}$$

Can we get back the sharp image from our blurry measurement? Examining the nullspace of B, denoted $N(B)$, will help us find out!

(a) Suppose we know that there are two sharp images, $\vec{x}_1$ and $\vec{x}_2$, that both result in the same blurry image $\vec{y}$.

$$\vec{y} = B\vec{x}_1 \qquad \vec{y} = B\vec{x}_2$$

Select all of the statements that are always true for any $\vec{x}_1$ and $\vec{x}_2$. Assume $\vec{y}$ is nonzero.

(1) $(4\vec{x}_1 + 10\vec{x}_2) \in N(B)$
(2) $(-8\vec{x}_1 + 8\vec{x}_2) \in N(B)$
(3) $(2\vec{x}_2 - 2\vec{x}_1) \in N(B)$
(4) $(2\vec{x}_2 - 18\vec{x}_1) \in N(B)$
(5) $\vec{x}_2 \in N(B)$

Answer: (2), (3)

(b) Suppose we know that

$$\begin{bmatrix} -3 \\ 5 \\ 3 \end{bmatrix} \in N(B) \qquad \text{and} \qquad \vec{x} = \begin{bmatrix} 13 \\ -11 \\ 10 \end{bmatrix} \text{ is a solution to } \vec{y} = B\vec{x}.$$

Find a different image $\vec{w}$, where $\vec{w} \neq \vec{x}$, that is guaranteed to satisfy the measurement (e.g. $\vec{y} = B\vec{w}$).

Answer: $\begin{bmatrix} 10 \\ -6 \\ 13 \end{bmatrix}$

Your friend suggests you try changing the nullspace of the blurring operator by applying an invertible linear transformation after capturing the image.

In the next two parts, you'll prove that

$$N(AB) = N(B)$$

for an invertible matrix A where $A, B \in \mathbb{R}^{n \times n}$.

(c) First we'll show that any vector $\vec{v}$ in the nullspace of B is also in the nullspace of AB. Complete the proof by filling in the question marks from the bank of options below. Each step should follow logically from the previous step, and each step can only be used once at most.

$\vec{v} \in N(B)$
?
?
?
$\vec{v} \in N(AB)$

Options:

A: $A\vec{v} = \vec{0}$	**B:** $A\vec{v}B = \vec{0}$	**C:** $B\vec{v} = \vec{0}$
D: $\vec{v} = B^{-1}\vec{0}$	**E:** $AB\vec{v} = A\vec{0}$	**F:** $BA\vec{v} = \vec{0}$
G: $A\vec{0} = \vec{v}$	**H:** $B^{-1}B\vec{v} = B^{-1}\vec{0}$	**I:** $B\vec{v} = A^{-1}\vec{0}$
J: $A^{-1}AB\vec{v} = A^{-1}\vec{0}$	**K:** $B^2\vec{v} = B\vec{0}$	**L:** $B\vec{v} = \vec{0}A^{-1}$
M: $AB\vec{v} = \vec{0}$	**N:** $A\vec{v} = B^{-1}\vec{0}$	**O:** $B\vec{0} = \vec{v}$

To enter your answer in Gradescope, type the 3 letters corresponding to your choices in order. For example, if your answer is the first row of choices (in order), type "ABC" in Gradescope. To leave a spot blank, type "?" in that spot. For example, "A?B" indicates that you are not making a choice for the second unknown. Remember, there will be no additional penalty for incorrect answers.

Answer: CEM

(d) Next, we'll show that any vector $\vec{u}$ in the nullspace of AB is also in the nullspace of B. Complete the proof by filling in the question marks from the bank of options below. Each step should follow logically from the previous step, and each step can only be used once at most.

$\vec{u} \in N(AB)$
?
?
?
?
$\vec{u} \in N(B)$

Options:

A: $B\vec{u} = \vec{0}A^{-1}$	**B:** $A^{-1}AB\vec{u} = A^{-1}\vec{0}$	**C:** $A\vec{u} = \vec{0}$
D: $AB\vec{u} = \vec{0}$	**E:** $B\vec{0} = \vec{u}$	**F:** $B^2\vec{u} = B\vec{0}$
G: $A\vec{0} = \vec{u}$	**H:** $B^{-1}B\vec{u} = B^{-1}\vec{0}$	**I:** $AB\vec{u} = A\vec{0}$
J: $\vec{u} = B^{-1}\vec{0}$	**K:** $B\vec{u} = \vec{0}$	**L:** $B\vec{u} = A^{-1}\vec{0}$
M: $A\vec{u} = B^{-1}\vec{0}$	**N:** $A\vec{u}B = \vec{0}$	**O:** $BA\vec{u} = \vec{0}$

To enter your answer in Gradescope, type the 4 letters corresponding to your choices in order. For example, "ABCD." To leave a spot blank, type "?" in that spot. For example, "A?BC" indicates that you are not making a choice for the second unknown. Remember, there will be no additional penalty for incorrect answers.

Answer: DBLK