EECS 16A Designing Information Devices and Systems I
Spring 2020 Lecture Notes Note 1

Overview

In this note, we introduce systems of linear equations, which arise frequently in modeling real-world problems. We begin with a tomography problem, wherein a system of linear equations naturally emerges via the measurements recorded. Using this as a motivating example, we introduce general systems of linear equations, and discuss their systematic solution using Gaussian elimination.

1.1 What is Linear Algebra?

- Linear algebra is the study of linear functions and linear equations, typically via their representation using vectors and matrices.
- A lot of objects in EECS can be treated as vectors and studied with linear algebra.
- Linearity is a good first-order approximation to the complicated real world.
- There exist good fast algorithms to do many of these manipulations in computers.
- Linear algebra concepts are an important tool for modeling the real world.

As you will see in the homeworks and labs, these concepts can be used to do many interesting things in real-world-relevant application scenarios. In the previous note, we introduced the idea that all information devices and systems (1) take some piece of information from the real world, (2) convert it to the electrical domain for measurement, and then (3) process these electrical signals. Because so many efficient algorithms exist that perform linear algebraic manipulations with computers, linear algebra is often a crucial component of this processing step.

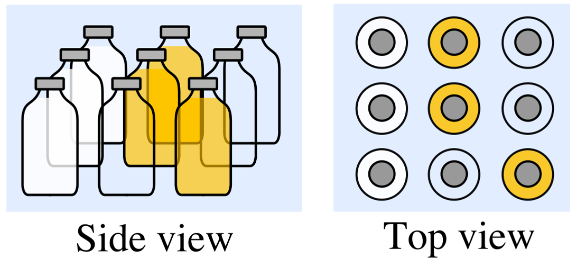
1.2 Application: Tomography

Throughout this course, we will motivate the introduction of concepts by considering a real-world application - this is the first one!

Tomography allows us to "see inside" a solid object, such as the human body or even the earth, by taking images section by section with a penetrating wave, such as X-rays. CT scans in medical imaging are perhaps the most famous such example — in fact, CT stands for "computed tomography."

Let's look at a specific toy example, using tomography to help with a (fairly unlikely!) real-world scenario.

A grocery store employee just had a truck load of bottles given to him. Each bottle is either empty, contains milk, or contains juice, and the bottles are packaged in boxes, with each box containing 9 bottles in a 3×3 grid. Inside a single box, it might look something like this:



If we choose symbols such that M represents milk, J represents juice, and O represents an empty bottle, we can represent the stack of bottles shown above as follows:

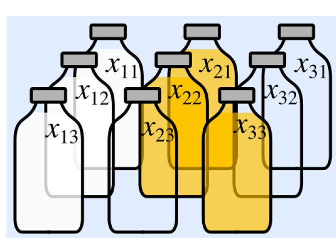
$$\begin{matrix} M & J & O \\ M & J & O \\ M & O & J \end{matrix} \tag{1}$$

Imagine that our grocer cannot see directly into the box, but needs to determine its contents using a light source and light sensor. How can we help him do this?

Let the light source emit light with a certain known intensity. As the light passes through a bottle, its intensity diminishes by an amount that depends on the contents of the bottle - milk absorbs 3 units of light, juice absorbs 2 units of light and an empty bottle absorbs 1 unit of light. The box itself does not affect the intensity of the light. After the light emitted exits the box, we can use our sensor to measure the final intensity, and so determine the amount of light absorbed by each bottle.

Thus, if we shine light in a straight line through some bottles within the box, we can determine the total amount of light absorbed by the bottles as the sum of the light absorbed by each bottle. For instance, in our specific example, shining a light from left to right would look like this, with each row observed to absorb 6 total units of light:

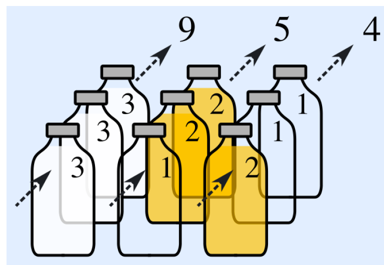
In order to deal with this more generally, let's assign variables to the amount of light absorbed by each bottle:



This means that x_{11} would be the amount of light the top left bottle absorbs, x_{21} would be the amount of light the top middle bottle absorbs, and so forth. Shining the light from left to right for our specific example gives the following equations:

$$x_{11} + x_{21} + x_{31} = 6 \tag{2}$$
$$x_{12} + x_{22} + x_{32} = 6 \tag{3}$$
$$x_{13} + x_{23} + x_{33} = 6 \tag{4}$$

Similarly, we could consider shining a light from bottom to top:



Which would give the following equations:

$$x_{13} + x_{12} + x_{11} = 9 \tag{5}$$
$$x_{23} + x_{22} + x_{21} = 5 \tag{6}$$
$$x_{33} + x_{32} + x_{31} = 4 \tag{7}$$

Thus, we now know how our to determine our observations given the contents of the box. But can we do the reverse? That is to say, given the amounts of light absorbed by each row and column of bottles, can we reconstruct the box's original contents?

From our above observations, one possible assignment of values to the x_{ij} (corresponding to the actual configuration of bottles) is

$$\begin{matrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{matrix} \tag{8}$$

However, the following assignment of values also works:

$$\begin{matrix} 3 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{matrix} \tag{9}$$

which corresponds to a different configuration of bottles within the box. In other words, our observations are not sufficient to **uniquely** identify the configuration of bottles within the box. This is a problem!

Intuitively, if we can't identify an object in real-life from a set of observations, we make more observations - the same principle seems to apply here. To get these additional observations, we could shine light at different angles through the box.

This brings up some very natural questions: Would shining light through the diagonals of the box would provide us with enough information? If not, how many different directions do we need to shine light through before we are certain of the configuration of bottles? Do some measurements provide us with more information than others do? What happens as we vary the number of bottles in our box?

In Module 1 of this course, we will develop the tools to answer all of these questions.

1.3 What is a System of Equations?

A **system of equations** is nothing but a collection of one or more equations, expressed in terms of **functions**, **unknowns** (also called **variables**) and **constant terms**. In particular, if we are given functions $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ (this notation means that each f_i is a real function of n real variables) and constants $b_i \in \mathbb{R}$ for each $i = 1, \ldots, m$, then we may write a corresponding system of equations

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= b_1 \\ f_2(x_1, x_2, \ldots, x_n) &= b_2 \\ \vdots \qquad & \quad \vdots \\ f_m(x_1, x_2, \ldots, x_n) &= b_m, \end{aligned}$$

where $x_1, \ldots, x_n$ are the unknowns. A **solution** to the linear system is an assignment of values to the unknowns $x_1, \ldots, x_n$ such that all equations are simultaneously satisfied. In other words, when solving a system of equations, we try to answer the following question: *for what values $x_1, \ldots, x_n$ will the above equalities be satisfied?* Consider, for example, the tomography problem of the previous section. In the tomography problem, the functions f_i could be expressed as weighted sums of the input variables. The resulting system of equations was an example of what is called a **system of linear equations**. The key word here is “linear”, which distinguishes such systems from more general systems of equations. This is the topic of the next section.

1.4 Systems of Linear Equations

1.4.1 Linear Equations

A system of equations is **linear** if each of the functions involved is *linear* in its variables. This statement seems a bit self-referential, but in fact it makes good sense once we define what it means for a function to be linear in its variables. To this end, we make the following definition:

Definition 1.1 (Linear Functions): A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is *linear* if for all real-valued $\alpha, \beta, y_1, \ldots, y_n, z_1, \ldots, z_n$, the following identity holds:

$$f(\alpha y_1 + \beta z_1, \alpha y_2 + \beta z_2, \ldots, \alpha y_n + \beta z_n) = \alpha f(y_1, \ldots, y_n) + \beta f(z_1, \ldots, z_n). \tag{10}$$

In words, a linear function satisfies the following two properties:

i) *Homogeneity*: scaling the input to the function scales the output by the same amount.

ii) *Superposition*: the function evaluated on the sum of two choices of input variables is equal to the sum of the function evaluated on each choice of input variables separately.

An extremely useful fact is that all linear functions can be represented as a weighted sum of the input variables. Let's state this important observation as a Theorem.

Theorem 1.1: If $f : \mathbb{R}^n \to \mathbb{R}$ is linear, then there exist coefficients $c_1, c_2, \ldots, c_n$ (i.e., real constants, not depending on the input to the function) such that

$$f(x_1, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \quad \text{for all } x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}. \tag{11}$$

The expression on the right hand side of (11) is called a **linear combination** of the x_i's. Hence, any linear function can be expressed as a linear combination of its inputs.

Why is this true? Well, the only thing we know at this point is the definition of linearity, so let's use it. In particular, we can start by cleverly rewriting $f(x_1, \ldots, x_n)$ as

$$f(x_1, \ldots, x_n) = f(\underbrace{x_1 \times 1 + 1 \times 0}_{x_1}, \underbrace{x_1 \times 0 + 1 \times x_2}_{x_2}, \underbrace{x_1 \times 0 + 1 \times x_3}_{x_3}, \ldots, \underbrace{x_1 \times 0 + 1 \times x_n}_{x_n}). \tag{12}$$

Conveniently, this is precisely in the form of (10) with $\alpha = x_1$, $\beta = 1$, $(y_1, y_2, \ldots, y_n) = (1, 0, \ldots, 0)$ and $(z_1, z_2, \ldots, z_n) = (0, x_2, \ldots, x_n)$. Thus, we apply the definition of linearity to conclude

$$f(x_1, \ldots, x_n) = x_1 f(1, 0, \ldots, 0) + f(0, x_2, \ldots, x_n). \tag{13}$$

We can apply the same trick again to see that

$$f(0, x_2, \ldots, x_n) = x_2 f(0, 1, 0, \ldots, 0) + f(0, 0, x_3, \ldots, x_n) \tag{14}$$

and so forth. Putting it all together, we have

$$f(x_1, \ldots, x_n) = x_1 f(1, 0, \ldots, 0) + x_2 f(0, 1, 0, \ldots, 0) + \cdots + x_n f(0, 0, \ldots, 0, 1). \tag{15}$$

Now, for each $i = 1, \ldots, n$, we define $c_i := f(0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the ith position. This is just a real number, not depending on $x_1, \ldots, x_n$, and therefore we have convinced ourselves that (11) is indeed true.

A **linear equation** is simply an equation of the form $f(x_1, \ldots, x_n) = b$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function, $b \in \mathbb{R}$ is a given constant, and the $x_1, \ldots, x_n$ are real-valued unknowns (i.e., variables).

Let's consider a few examples. For variables x and y, the statement

$$5 \times x + 6 \times y = 7 \tag{16}$$

is a linear equation. Indeed, the left hand side is a linear function of x and y.

In contrast, the equation

$$y \times y = 5 \tag{17}$$

is not a linear equation since the function $f(y) = y^2$ is not linear (it is a good exercise to convince yourself of this fact).

Observe that equations such as

$$8x = 4y,$$

or

$$8x - 4y = 0,$$

or

$$2x - y = 0$$

are all examples of linear equations. Notice that the constant term in a linear equation is allowed to be 0.

1.4.2 Affine Functions

What about functions like

$$f_3(x) = 2x + 1, \quad x \in \mathbb{R}?$$

Plotting this function, we see that it is a line. But it doesn't seem to fit into the form $f(x) = cx$, so is it linear? A simple check, if we're ever unsure about the behavior of a function, is to plug in some simple input values and see how the output behaves. Let's do that here, for $x = 1$ and $x = 2$. We see that

$$f_3(1) = 3 \text{ and } f_3(2) = 5,$$

so doubling the input value from 1 to 2 changes the output by a factor of $5/3$. Thus, this function is not linear, *even though* it describes the equation of a line. This motivates the following definition: A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be an **affine function** if it can be written in the form

$$g(x_1, \dots, x_n) = f(x_1, \dots, x_n) + c_0 \quad \text{for all } x_1 \in \mathbb{R}, \dots, x_n \in \mathbb{R},$$

for some linear function $f : \mathbb{R}^n \to \mathbb{R}$ and constant term $c_0 \in \mathbb{R}$. By applying Theorem 1.1, we conclude that any affine function can be written as

$$g(x_1, \dots, x_n) = c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

Notice that the definition of affine functions includes all linear functions (by setting the scalar constant to 0), so every linear function is also affine, though not vice-versa. Nevertheless, *a system of equations involving all affine functions is still a system of linear equations.* (why?)

These definitions mean that while all functions describing a line can be shown to be affine, not all of them are linear. This has the unfortunate consequence that, in informal conversation, *affine* functions may be called *linear*, since both describe a line. This usage, though common, is **wrong**, as we saw with the example of f_3.

1.4.3 Expressing Systems of Linear Equations

In its most general form, a system of m linear equations involving n variables can be written as

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= b_1 \\ f_2(x_1, x_2, \ldots, x_n) &= b_2 \\ \vdots \quad\quad & \quad \vdots \\ f_m(x_1, x_2, \ldots, x_n) &= b_m, \end{aligned}$$

where each f_i is a linear function, and b_i is a constant. From Theorem 1.1, we know that each linear function f_i can be expressed as a linear combination:

$$f_i(x_1, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

for appropriate coefficients $a_{i1}, \ldots, a_{in}$. Hence, any system of m linear equations in n variables can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\quad \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n. \end{aligned} \tag{18}$$

Generally, the a_{ij}'s and b_i's are fixed constants, and we aim to find a solution $x_1, \ldots, x_n$ to the above system of equations (i.e., a choice of $x_1, \ldots, x_n$ that satisfies each of the above equations). Indeed, a good exercise is to note that the tomography problem fits into this framework. There, the a_{ij}'s are determined by the configuration of bottles we shine the light through, and the b_i's are the light intensity we measure at the sensor. Moreover, each linear equation corresponds to a different measurement.

1.5 Augmented Matrices

Observe that the general expression for a system of linear equations given in (18) is a bit tedious to write. To address this, we introduce a notational device known as an **augmented matrix** to simplify things. The importance of this notational device goes beyond convenience, it will allow us to view the process of solving systems of equations as a sequence of simple modifications of the corresponding augmented matrix. More specifically, we arrange the coefficients a_{ij} and constants b_i in the system (18) into an array as follows:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_n \end{array}\right] \tag{19}$$

In the above, the entry in row i and column $j \leq n$ corresponds to the coefficient preceding variable x_j in linear equation i in (18). Similarly, the entry in row i and column n is b_i, corresponding to the constant in linear equation i. This representation of a linear system is known as the **augmented matrix representation**. An interesting thing to notice about this representation is that the symbols corresponding to our unknowns have vanished entirely! This reinforces what we know already: the label we assign to any given variable is immaterial (e.g., we can call our variables $x_1, \ldots, x_n$ or $y_1, \ldots y_n$ or $u, v, w, \ldots, z$, the specific choice of label doesn't matter).

When representing a system of linear equations as an augmented matrix, we always draw a line before the last column as done above, since that column contains the constants, not the coefficients, corresponding to our system of linear equations. This is important: it distinguishes an augmented matrix (a notational device used to represent a system of equations) from a matrix (a mathematical object, to be introduced soon).

Let us again consider the tomography example of the first section. There, we obtained the system of linear equations

$$\begin{aligned}
x_{11} + x_{21} + x_{31} &= 6 \\
x_{12} + x_{22} + x_{32} &= 6 \\
x_{13} + x_{23} + x_{33} &= 6 \\
x_{13} + x_{12} + x_{11} &= 9 \\
x_{23} + x_{22} + x_{21} &= 5 \\
x_{33} + x_{32} + x_{31} &= 4.
\end{aligned} \tag{20}$$

Recall that the first three equations of the above system came from measuring the light absorbed through each row of bottles, while the second three equations came from measurements of the columns. Don't be fooled by the double subscripts on the variables in form (20). By introducing coefficients equal to zero as follows:

$$\begin{aligned}
1 \times x_{11} + 0 \times x_{12} + 0 \times x_{13} + 1 \times x_{21} + 0 \times x_{22} + 0 \times x_{23} + 1 \times x_{31} + 0 \times x_{32} + 0 \times x_{33} &= 6 \\
0 \times x_{11} + 1 \times x_{12} + 0 \times x_{13} + 0 \times x_{21} + 1 \times x_{22} + 0 \times x_{23} + 0 \times x_{31} + 1 \times x_{32} + 0 \times x_{33} &= 6 \\
0 \times x_{11} + 0 \times x_{12} + 1 \times x_{13} + 0 \times x_{21} + 0 \times x_{22} + 1 \times x_{23} + 0 \times x_{31} + 0 \times x_{32} + 1 \times x_{33} &= 6 \\
1 \times x_{11} + 1 \times x_{12} + 1 \times x_{13} + 0 \times x_{21} + 0 \times x_{22} + 0 \times x_{23} + 0 \times x_{31} + 0 \times x_{32} + 0 \times x_{33} &= 9 \\
0 \times x_{11} + 0 \times x_{12} + 0 \times x_{13} + 1 \times x_{21} + 1 \times x_{22} + 1 \times x_{23} + 0 \times x_{31} + 0 \times x_{32} + 0 \times x_{33} &= 5 \\
0 \times x_{11} + 0 \times x_{12} + 0 \times x_{13} + 0 \times x_{21} + 0 \times x_{22} + 0 \times x_{23} + 1 \times x_{31} + 1 \times x_{32} + 1 \times x_{33} &= 4.
\end{aligned}$$

Having expressed the problem in standard form, we may immediately write its representation in terms of an augmented matrix:

$$\left[\begin{array}{ccccccccc|c}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 6 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 6 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 6 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4
\end{array}\right]. \tag{21}$$

1.6 Solving Systems of Linear Equations – An example

We may already be able to solve linear systems in a so-called "ad-hoc" manner - manipulating equations according to our intuition until we get a solution or have convinced ourselves that no solution exists. Let's remind ourselves of how we could do that.

Consider the linear system with unknowns x and y:

$$\begin{aligned}5x+6y&=40\\8x+9y&=61.\end{aligned}$$

To solve it, one way would be to use the first equation to express y in terms of x, and then substitute into the second to obtain a single linear equation involving only x. Rearranging the first equation, we find that

$$\begin{aligned}&& 5x+6y&=40\\ \implies && 6y&=40-5x\\ \implies && y&=\frac{20}{3}-\frac{5}{6}x.\end{aligned}$$

Now, substituting into the second equation, we obtain

$$\begin{aligned}&& 8x+9y&=61\\ \implies && 8x+9\left(\frac{20}{3}-\frac{5}{6}x\right)&=61\\ \implies && \frac{1}{2}x+60&=61\\ \implies && \frac{1}{2}x&=1\\ \implies && x&=2.\end{aligned}$$

Finally, substituting this value for x into our equation for y in terms of x, we find

$$\begin{aligned}y&=\frac{20}{3}-\frac{5}{6}x\\&=\frac{20}{3}-\frac{5}{6}\cdot 2\\&=5.\end{aligned}$$

This ad-hoc, substitution-based approach worked well for this small system. However, as we start to consider larger systems, potentially with thousands or even millions of variables, manually solving them quickly becomes infeasible.

We need to develop a systematic approach to solve linear systems that generalizes well as the numbers of equations and variables increases. After developing this approach, we should be able to prove (at least to a certain level of rigor) that our approach is guaranteed to work whenever a solution to a linear system exists.

1.7 Gaussian Elimination

Gaussian elimination is an **algorithm** (a sequence of programmatic steps) that accomplishes this task. Specifically, it can be used to solve any arbitrarily large system of linear equations, or decide that no solution exists. Gaussian elimination isn't the *only* algorithm that does this (for instance, we could try writing an algorithm that formalizes the substitution-based approach that we tried above), but it's pretty good!

Side note: Gaussian elimination was named after Carl Friedrich Gauss, a German mathematician from the 18th century. Despite being its namesake, he did not invent it, though he contributed to its development. As it turns out, Gaussian elimination was initially developed in China over 2000 years ago!

In Europe, Gaussian elimination was refined over the course of 200 years by mathematicians including Newton, Rolle, and Gauss. To quote Newton,

> And you are to know, that by each Æquation one unknown Quantity may be taken away, and consequently, when there are as many Æquations and unknown Quantities, all at length may be reducæd into one, in which there shall be only one Quantity unknown.

To read about how it evolved, check out `https://www.ams.org/notices/201106/rtx110600782p.pdf`!

1.7.1 Example

Before we look at a precise formulation of Gaussian elimination, let's look at an example of how it works, to build our intuition. Specifically, let's try solving the following example using Gaussian elimination:

$$\begin{aligned} 5x+6y+z &= 43 \quad (1) \\ 8x+9y+2z &= 67 \quad (2) \\ x+y+4z &= 19. \quad (3) \end{aligned}$$

Intuitively, the basic idea behind Gaussian elimination is to use each equation to *eliminate* one variable from all the subsequent equations, until we end up with an equation with just one unknown, which we can directly solve. (we'll make this intuition more rigorous in a moment). Let's try using the first equation to eliminate one of our unknowns - say, x. It would make sense to first **multiply** the first equation by $1/5$, in order to remove the coefficient of 5 in front of x. Thus, we obtain the equivalent[1] system of equations

$$\begin{aligned} x+\frac{6}{5}y+\frac{1}{5}z &= \frac{43}{5} \quad (1') \\ 8x+9y+2z &= 67 \quad (2) \\ x+y+4z &= 19. \quad (3) \end{aligned}$$

Now, we will try to eliminate the variable x in (2) and (3). One way would be to write x in terms of y and substitute, like we did earlier. Instead, however, Gaussian elimination requires us to **add multiples** of (1') to (2) and (3), in order to accomplish the same goal (we'll see why we're currently trying to avoid substitution in a moment, when we make this process more rigorous).

Let's try eliminating x from (2) first. What multiple of (1') would be best to use? Well, since (1') has an x term, and we'd like it to cancel out with the $8x$ term from (2), it makes sense to multiply (1') by -8 and add

[1] Equivalence here is meant to mean that each system of equations has the same set of solutions.

it to (2) to produce (2') below:

$$\begin{aligned} & (8x+9y+2z) - 8\cdot\left(x+\frac{6}{5}y+\frac{1}{5}z\right) = 67 - 8\cdot\frac{43}{5} \\ \implies & \frac{-3}{5}y+\frac{2}{5}z = -\frac{9}{5} \qquad (2') \end{aligned}$$

Similarly, it would make sense to simply subtract (1') itself from (3) (equivalently, to multiply (1') by -1 and add it to (3)), since both (3) and (1') have an x term with a coefficient of one. This produces

$$\begin{aligned} & (x+y+4z) - \left(x+\frac{6}{5}y+\frac{1}{5}z\right) = 19 - \frac{43}{5} \\ \implies & -\frac{1}{5}y+\frac{19}{5}z = \frac{52}{5}. \qquad (3') \end{aligned}$$

Now, observe that (2') and (3') together form a linear system with just two unknowns: y and z, since x has been eliminated from the latter two equations. Now, let's see if we can repeat this process, using (2') to eliminate y from (3'). First, as before, we should simplify things by scaling (2') to remove the coefficient of y. Multiplying (2') by $-5/3$, we obtain

$$y - \frac{2}{3}z = 3. \quad (2'')$$

Again, we'd like to add some scalar multiple of (2") to (3'), in order to eliminate y from (3'). Since (3') has a $(-1/5)y$ term, it makes sense to multiply (2") by $1/5$ and add it to (3'), which produces

$$\begin{aligned} & -\frac{1}{5}y+\frac{19}{5}z+\frac{1}{5}\left(y-\frac{2}{3}z\right) = \frac{52}{5}+\frac{1}{5}\cdot 3 \\ \implies & \frac{11}{3}z = 11. \qquad (3'') \end{aligned}$$

Now, (3") is a simple linear equation in only one variable, which we know how to immediately solve. Awesome! Rearranging a little, we see that

$$z = 3.$$

When we did our initial example above, after solving for x, we substituted it into an equation for y in terms of x. Can we do something similar here? Specifically, do we have an equation for x or y in terms of just z?

Notice that the equation we used to eliminate y (Eq. 2") had no variables "before" y, since they were eliminated, and doesn't even have a coefficient for y. Pulling all terms except for y onto the right-hand-side of the equality, we obtain

$$y = 3 + \frac{2}{3}z.$$

Perfect! Substituting in $z = 3$, we find that

$$y = 3 + \frac{2}{3}\cdot 3 = 5.$$

Can we do the same thing again? Well, when we eliminated x using (1'), we again needed an equation with a coefficient of one for x. Pulling all the terms in (1') except for x to the right-hand-side, we obtain

$$x = \frac{43}{5} - \frac{6}{5}y - \frac{1}{5}z.$$

Again, this is looking pretty good. Substituting in our known values for y and z, we find that

$$x = \frac{43}{5} - \frac{6}{5} \cdot 5 - \frac{1}{5}z = 2,$$

so we've solved for all of our unknowns using Gaussian elimination. Awesome!

1.7.2 Steps of Gaussian Elimination

Let's take a moment to reflect on the approach we just used.

- First, we selected an equation involving x (possibly with some coefficient) and scaled it to make the x coefficient unity.
- Then, we added multiples of this equation from all the other equations to eliminate x, producing a system with one fewer unknown and one fewer equation.
- We then repeated the first two steps until we arrived at an equation with exactly one unknown, which we could solve directly.
- Finally, we substituted the known value of the final unknown into a previous equation to recover the last two unknowns, and continued substituting until we recovered all of our unknowns!

These are the key steps of Gaussian elimination. The first three steps are known as **row reduction**, and the final step is known as **back-substitution**.

Now that we know how to perform the steps of Gaussian elimination for systems where a solution is known to exist, it is important to ask ourselves *why* these steps work. In particular, even if we have some intuition for why they work in cases when a system of equations has a unique solution, we'd like to show that these steps remain valid even when working with a system with zero or infinitely many solutions.

1.7.3 Operations

The key idea behind Gaussian elimination is that of "invertible" operations. As we manipulate our equations, we want to preserve their set of solutions. In particular, we neither want to *introduce* new solutions in the process, nor to *remove* potential solutions. To do so, we use three operations that we are certain will *never* change the solution set of a system, and then apply these operations repeatedly in order to solve a linear system. By only applying these operations, we can be confident that our approach will never yield a wrong answer, since the solution set of our system is preserved throughout.

These operations, which we have just seen, are as follows:

1. **Multiplying an equation by a *nonzero* scalar constant.** For instance, if we have the equation

$$2 \times a + 3 \times b = 4,$$

we can multiply it by the nonzero scalar -2 to obtain

$$-4 \times a + (-6) \times b = -8.$$

Expressed as a single operation, we can write

$$\begin{aligned} & 2 \times a + 3 \times b = 4 \\ \implies \quad & -4 \times a + (-6) \times b = -8. \end{aligned}$$

Why does this operation preserve all the solutions to a system? Well, consider any particular solution that satisfies the first equation. Clearly, it still satisfies the second equation, so this operation has not removed any potential solutions.

But does it introduce a new solution? Consider any particular solution to the second equation. Notice that we can multiply the second equation by the *reciprocal* of our original nonzero scalar multiplier, to obtain the first equation. Thus, this particular solution will also satisfy the first equation. In other words, no solution exists that satisfies the second equation, but not the first. Consequently, the second equation is not only *implied by*, but also *implies* the first equation.

When each of two equations imply the other, we say that they are *equivalent*, since replacing one with the other does not change their solution set. Notice that, to obtain equivalence, we had to restrict our multiplication to one by a *nonzero* scalar, not an arbitrary scalar. Otherwise, we would not be able to obtain the first equation from the second (since the reciprocal of zero is undefined), so we would only obtain a one-way implication, not two-way equivalence.

2. **Adding a scalar constant multiple of one equation to another.** For instance, if we have the equations

$$\begin{aligned} 5 \times a + 6 \times b &= 7 \qquad (1) \\ 8 \times a + 9 \times b &= 10, \quad (2) \end{aligned}$$

we can multiply the second equation by the scalar 3 and add it to the first, to obtain the new system

$$\begin{aligned} 29 \times a + 33 \times b &= 37 \qquad (3) \\ 8 \times a + 9 \times b &= 10, \quad (2) \end{aligned}$$

Clearly, observe that any solution to the first system will also be a solution to the second, since the first system of equations implies the second. But is the reverse true? Well, observe that equation (1) can be recovered by taking equation (3) and subtracting our scalar (in this case, 3) multiplied by equation (2). In other words, our second system is, not only implied by, but also implies the first system, so it does not introduce any new solutions. Thus, replacing the first system with the second does not change the solution set of our linear system, so this operation is valid.

3. **Swapping two equations.** We have not yet seen when we need to swap two equations (though we will in Example 1.3), but it is clear that the solution set of a linear system of equations does not depend on the order of equations! Therefore, this final operation is clearly valid.

Now we have developed these three operations, we can repeatedly use them in a structured manner to solve arbitrary systems of linear equations, as will be illustrated in the following examples:

Example 1.1 (System of 2 equations): Consider the following system of two equations with two variables:

$$\begin{array}{rcrcrl} x & - & 2y & = & 1 & \quad (1) \\ 2x & + & y & = & 7 & \quad (2) \end{array}$$

We would like to find an explicit formula for x and y, but the presence of both x and y in each of the equations prevents this. If we can eliminate a variable from one of the equations, we can get an explicit formula for the remaining variable. To eliminate x from (Eq. 2), we can subtract 2 times (Eq. 1) from (Eq. 2) to obtain a new equation, (Eq. $2'$):

$$\begin{array}{rcrcrl} 2x & + & y & = & 7 & \\ -2\times(x & - & 2y & = & 1) & \\ & & \Downarrow & & & \\ 2x & + & y & = & 7 & \\ -2x & + & 4y & = & -2 & \\ & & \Downarrow & & & \\ & & 5y & = & 5 & \quad (2') \end{array}$$

Scaling (Eq. 1) by the amount that x is scaled in (Eq. 2) allows us to cancel the x term. As a result, we can replace (Eq. 2) with (Eq. $2'$) to rewrite our system of equations as:

$$\begin{array}{rrcrcrl} & x & - & 2y & = & 1 & \quad (1) \\ \text{(Eq. 2)} - 2 \times \text{(Eq. 1)}: & & & 5y & = & 5 & \quad (2') \end{array}$$

From here, we can divide both sides of (Eq. $2'$) by 5 to see that $y = 1$. We will call this (Eq. $2''$). Next, we would like to solve for x. It would be natural to proceed by substituting $y = 1$ into (Eq. 1) to solve directly for x, and doing this will certainly give you the correct result. However, our goal is to find an *algorithm* for solving systems of equations. This means that we would like to be able to repeat the same sequence of operations over and over again to come to the solution. Recall that to cancel x in (Eq. 2), we:

1. **Scaled** (Eq. 1) by a factor of 2.
2. **Subtracted** (Eq. 1) from (Eq. 2).

To solve for x, we would like to eliminate y from (Eq. 1) using a similar process of scaling and subtracting. Because y is scaled by a factor of -2 in (Eq. 1), we can scale (Eq. $2''$) by -2 and subtract it from (Eq. 1) to cancel the y term. Doing so gives:

$$\begin{array}{rrrcrl} \text{(Eq. 1)} + 2 \times \text{(Eq. } 2''\text{)}: & x & & = & 3 & \quad (1') \\ \text{(Eq. } 2'\text{)} \times \frac{1}{5}: & & y & = & 1 & \quad (2'') \end{array}$$

Soon we will generalize this technique so that it can be extended to any number of equations. Right now we will use an example with 3 equations to help build intuition.

Example 1.2 (System of 3 equations): Suppose we would like to solve the following system of 3 equations:

$$\begin{array}{rcrcrcrl} x & - & y & + & 2z & = & 1 & \quad (1) \\ 2x & + & y & + & z & = & 8 & \quad (2) \\ -4x & + & 5y & & & = & 7 & \quad (3) \end{array}$$

As in the 2 equation case, our first step is to eliminate x from all but one equation by adding or subtracting scaled versions of the first equation from the remaining equations. Because x is scaled by 2 and -4 in (Eq. 2) and (Eq. 3) (respectively), we can multiply (Eq. 1) by these factors and subtract it from the corresponding equations:

$$\begin{array}{rrcrcrcrl} & x & - & y & + & 2z & = & 1 & (1) \\ \text{(Eq. 2)} - 2\times \text{(Eq. 1)}: & & & 3y & - & 3z & = & 6 & (2') \\ \text{(Eq. 3)} + 4\times \text{(Eq. 1)}: & & & y & + & 8z & = & 11 & (3') \end{array}$$

Next, we would like to eliminate y from (Eq. $3'$). First, we can divide (Eq. $2'$) by 3 such that y is scaled by 1:

$$\begin{array}{rrcrcrcrl} & x & - & y & + & 2z & = & 1 & (1) \\ \text{(Eq. } 2')/3: & & & y & - & z & = & 2 & (2'') \\ & & & y & + & 8z & = & 11 & (3') \end{array}$$

Now, since y is also scaled by 1 in (Eq. $3'$), we can subtract (Eq. $2''$) from (Eq. $3'$) to get a formula[2] with only z:

$$\begin{array}{rrcrcrcrl} & x & - & y & + & 2z & = & 1 & (1) \\ & & & y & - & z & = & 2 & (2'') \\ \text{(Eq. } 3') - \text{(Eq. } 2''): & & & & & 9z & = & 9 & (3'') \end{array}$$

Dividing (Eq. $3''$) by 9 gives an explicit formula for z:

$$\begin{array}{rrcrcrcrl} & x & - & y & + & 2z & = & 1 & (1) \\ & & & y & - & z & = & 2 & (2'') \\ \text{(Eq. } 3'')/9: & & & & & z & = & 1 & (3''') \end{array}$$

At this point, we can see that our system of equations has a “triangular" structure — all three variables are contained in (Eq. 1), two are in (Eq. $2''$), and only z remains in (Eq. $3'''$). If we look back to the previous example with 2 equations, we obtained a similar result after eliminating x from (Eq. 2):

System of 2 Equations	System of 3 Equations
$\begin{array}{rcrcr} x & - & 2y & = & 1 \\ & & 5y & = & 5 \end{array}$	$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 1 \\ & & y & - & z & = & 2 \\ & & & & z & = & 1 \end{array}$

This similarity is not coincidental, but a direct result of the way in which we successively eliminate variables in our algorithm moving left to right. To understand why it is useful to have our system of equations in this format, we will now proceed to solve for the remaining variables in this 3-equation example. First, we would like to eliminate z from (Eq. 1) and (Eq. $2''$). As usual, we can accomplish this by scaling (Eq. $3'''$) by the

[2] At this point we have made a decision in our algorithm to eliminate y from (Eq. $3'$) but *not* (Eq. 1). The motivation for this might not be completely evident now, but approaching it this way can be more computationally efficient for certain systems of linear equations — typically if the system has an infinite number of solutions or no solutions.

amount z is scaled in (Eq. 1) and (Eq. 2$''$) and subtracting this from these equations:

$$\begin{array}{rrcrcrcr}
\text{(Eq. 1)} -2\times \text{(Eq. 3}'''\text{):} & x & - & y & & = & -1 & (1') \\
\text{(Eq. 2}''\text{)} + \text{(Eq. 3}'''\text{):} & & & y & & = & 3 & (2''') \\
& & & & z & = & 1 & (3''')
\end{array}$$

Finally, by adding (Eq. 2$'''$) to (Eq. 1$'$), we can find the solution:

$$\begin{array}{rrrrcrr}
\text{(Eq. 1}'\text{)} + \text{(Eq. 2}'''\text{):} & x & & & = & 2 & (1'') \\
& & y & & = & 3 & (2''') \\
& & & z & = & 1 & (3''')
\end{array}$$

After obtaining an explicit equation for z using a repetitive process of scaling and subtraction, we were able to obtain an explicit equation for y, and then x, using this same process — this time propagating equations upwards instead of downwards.

So far, the two operations we've previously encountered seem to be sufficient to solve every system of equations we've encountered. Are there any other operations we might need to perform in addition to scaling and adding/subtracting equations?

Example 1.3 (System of 3 equations): Suppose we would like to solve the following system of 3 equations:

$$\begin{array}{rcrcrcrr}
 & & 2y & + & z & = & 1 & (1) \\
2x & + & 6y & + & 4z & = & 10 & (2) \\
x & - & 3y & + & 3z & = & 14 & (3)
\end{array}$$

As in the 2 equation case, our first step is to eliminate x from all but one equation. Since x is the first variable to be eliminated, we want the equation containing it to be at the top. However, the first equation does not contain x. To solve this problem, we **swap** the first two equations. Clearly, swapping two equations does not change the system's solution set, so we will obtain the equivalent linear system:

$$\begin{array}{rrcrcrcrr}
\text{(Eq. 2):} & 2x & + & 6y & + & 4z & = & 10 & (1') \\
\text{(Eq. 1):} & & & 2y & + & z & = & 1 & (2') \\
& x & - & 3y & + & 3z & = & 14 & (3)
\end{array}$$

Now we can proceed as usual, dividing the first equation by 2, and then subtracting this from (Eq. 3$'$) to eliminate x:

$$\begin{array}{rrcrcrcrr}
\text{(Eq. 1}'\text{)} / 2\text{:} & x & + & 3y & + & 2z & = & 5 & (1'') \\
& & & 2y & + & z & = & 1 & (2') \\
\text{(Eq. 3)} - \text{(Eq. 1}''\text{):} & & - & 6y & + & z & = & 9 & (3')
\end{array}$$

Now there is only one equation containing x. Of the remaining two equations, we want only one of them to contain y, so we can divide (Eq. 2$'$) by 2 and then add 6 times (Eq. 2$''$) to (Eq. 3$'$):

$$\begin{array}{rrcrcrcrr}
& x & + & 3y & + & 2z & = & 5 & (1'') \\
\text{(Eq. 2}'\text{)} / 2\text{:} & & & y & + & \frac{1}{2}z & = & \frac{1}{2} & (2'') \\
\text{(Eq. 3}'\text{)} + 6\times \text{(Eq. 2}''\text{):} & & & & & 4z & = & 12 & (3'')
\end{array}$$

Now, we have the "triangular" structure from the previous examples. To proceed, we can divide the last equation by 4 to solve for $z = 3$, and use this to eliminate z from the remaining equations:

$$\begin{array}{rrcrcrcrl}
\text{(Eq. } 1'') - 2\times \text{ (Eq. } 3'''): & x & + & 3y & & & = & -1 & (1''') \\
\text{(Eq. } 2'') - \frac{1}{2}\times \text{ (Eq. } 3'''): & & & y & & & = & -1 & (2''') \\
\text{(Eq. } 3'')\ /\ 4: & & & & & z & = & 3 & (3''')
\end{array}$$

Finally, we can subtract 3 times (Eq. $2'''$) from (Eq. $1'''$) to solve for x:

$$\begin{array}{rrcrcrcrl}
\text{(Eq. } 1''') - 3\times \text{ (Eq. } 2'''): & x & & & & & = & 2 & (1'''') \\
& & & y & & & = & -1 & (2''') \\
& & & & & z & = & 3 & (3''')
\end{array}$$

1.7.4 Gaussian Elimination with Augmented Matrices

Now, let's try to apply our previous approach for solving linear equations on the augmented matrix representation of a linear system.

For convenience, rather than using the system of equations presented above, let's look at the simpler system, we saw in the previous note can be represented as an augmented matrix:

$$\begin{bmatrix} 5x & + & 3y & = & 5 \\ -4x & + & y & = & 2 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 5 & 3 & 5 \\ -4 & 1 & 2 \end{array}\right]$$

In the examples we have seen, there are three basic operations that we can perform to a system of equations, that we know will preserve the solution set of the associated system of linear equations. Let's see how they work when applied to the augmented matrix representation of a system of linear equations:

1. Multiplying a row by a nonzero scalar. For example, we can multiply the first row by 2:

$$\begin{bmatrix} 10x & + & 6y & = & 10 \\ -4x & + & y & = & 2 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 10 & 6 & 10 \\ -4 & 1 & 2 \end{array}\right]$$

2. Swapping rows. For example, we swap the 2 rows:

$$\begin{bmatrix} -4x & + & y & = & 2 \\ 5x & + & 3y & = & 5 \end{bmatrix} \qquad \left[\begin{array}{cc|c} -4 & 1 & 2 \\ 5 & 3 & 5 \end{array}\right]$$

3. Adding a scalar multiple of a row to another row. For example, we can modify the second row by adding 2 times the first row to the second:

$$\begin{bmatrix} 5x & + & 3y & = & 5 \\ 6x & + & 7y & = & 12 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 5 & 3 & 5 \\ 6 & 7 & 12 \end{array}\right]$$

The above three operations on an augmented matrix are called **elementary row operations**.

In what follows, we use the term **leading entry** of a row to describe the first (i.e., leftmost) nonzero entry in that row. We shall now describe how to apply Gaussian elimination to an augmented matrix of the form

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_n \end{array}\right], \tag{22}$$

which describes a system of m linear equations with n unknowns.

Gaussian Elimination Algorithm for Augmented Matrix (22).

Step 1: For $i = 1, 2, \dots, m$

a) If necessary, swap row i with a row below it, so that the leading entry in row i is as far left as possible.

b) Rescale row i so that its leading entry is equal to 1.

c) For rows $j = i+1, \dots m$, add to row j a scalar multiple of row i, so that the leading entry of row i has all zeros below it.

Step 2: For $i = m, m-1, \dots, 1$

a) Add to each row $j = 1, 2, \dots, i-1$ a scalar multiple of row i so that the leading entry of row i has all zeros above it.

Observe that at the conclusion of Step 1, we are left with a matrix satisfying the following three properties:

- All nonzero rows are above all zero rows.
- The leading entries of a non-zero row is always to the right of the leading entries of the row above it.
- All leading entries of non-zero rows are equal to 1.

A matrix satisfying the above three properties is said to be in **row echelon form**. Hence, Step 1 of the Gaussian Elimination algorithm reduces the augmented matrix to row echelon form. For illustrative purposes, the following represents a matrix in row echelon form

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{23}$$

where $*$ denotes that any value may be entered in that location.

Step 2 corresponds to the “back-substitution" of variables performed in the previous examples. At the conclusion of Step 2, we are left with a matrix satisfying the following two properties:

- The matrix is in row echelon form.

- Each leading entry of a nonzero row is the only nonzero entry in its column.

Such a matrix is said to be in **reduced row echelon form**, sometimes abbreviated (especially in programming) as `rref`. As another illustration, the following represents a matrix in reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{24}$$

By construction, the Gaussian elimination algorithm always results in a matrix that is in reduced row echelon form. Once an augmented matrix is reduced to reduced row echelon form, variables corresponding to columns containing leading entries are called **basic variables**, and the remaining variables are called **free variables**. For example, if we consider the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{25}$$

then it corresponds to the system of equations

$$\begin{array}{ccccl} x_1 & & +2x_3 & & = 3 \\ & x_2 & +4x_3 & & = 5 \\ & & & x_4 & = -8. \end{array} \tag{26}$$

In this system of equations, x_1, x_2, x_4 are all basic variables, and x_3 is a free variable. As will be discussed shortly, the distinction between basic and free variables allows us to characterize all solutions to the system of linear equations (if any exist!).

1.7.4.1 Gaussian Elimination Examples

Example 1.4 (Equations with exactly one solution):

$$\begin{bmatrix} 2x & + & 4y & + & 2z & = & 8 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 4 & 2 & 8 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right]$$

First, divide row 1 by 2, the scaling factor on x in the first equation.

$$\begin{bmatrix} x & + & 2y & + & z & = & 4 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right]$$

To eliminate x from the two remaining equations, subtract row 1 from row 2 and 3.

$$\begin{bmatrix} x & + & 2y & + & z & = & 4 \\ & - & y & & & = & 2 \\ & - & 3y & - & 2z & = & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & -3 & -2 & 0 \end{array}\right]$$

To ensure y is scaled by 1 in the second equation, multiply row 2 by -1. Then, to eliminate y from the final equation, subtract -3 times row 2 from row 3.

$$\begin{bmatrix} x & + & 2y & + & z & = & 4 \\ & & y & & & = & -2 \\ & & & - & 2z & = & -6 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -2 & -6 \end{array}\right]$$

To scale z by 1 in the final equation, divide row 3 by -2.

$$\begin{bmatrix} x & + & 2y & + & z & = & 4 \\ & & y & & & = & -2 \\ & & & & z & = & 3 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Notice that our matrix is now in *row echelon* form, since the leading coefficient of each nonzero row is to the right of the leading coefficient of the row above it. However, it is not yet in *reduced row echelon form*, since the second and third columns, which each contain an element in the pivot position of a row, have a nonzero element in another row.

Continuing, we then subtract row 3 from row 1 to eliminate z from the first equation.

$$\begin{bmatrix} x & + & 2y & & & = & 1 \\ & & y & & & = & -2 \\ & & & & z & = & 3 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Finally, subtract 2 times row 2 from row 1 to obtain an explicit equation for all variables.

$$\begin{bmatrix} x & & & & & = & 5 \\ & & y & & & = & -2 \\ & & & & z & = & 3 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Observe that our matrix is now in *reduced row echelon form*, since in addition to still being in row echelon form, each column with an element in pivot position (which, in this case, are all the columns) has only one nonzero element, which equals 1 and is in the pivot position of a row.

This system of equations has a unique solution — x, y, and z can take on only *one* value in order for each equation to be true.

Example 1.5 (Equations with an infinite number of solutions):

$$\begin{bmatrix} x & + & y & + & 2z & = & 2 \\ & & y & + & z & = & 0 \\ 2x & + & y & + & 3z & = & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 4 \end{array}\right]$$

To eliminate x from the third equation, subtract 2 times row 1 from row 3.

$$\begin{bmatrix} x & + & y & + & 2z & = & 2 \\ & & y & + & z & = & 0 \\ & - & y & - & z & = & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

To eliminate y from the third equation, add row 2 to row 3.

$$\begin{bmatrix} x & + & y & + & 2z & = & 2 \\ & & y & + & z & = & 0 \\ & & & & 0 & = & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

At this point, the third equation no longer contains z so we cannot "eliminate" it. We can, however, proceed by eliminating y from the first equation. To do this, subtract row 2 from row 1.

$$\begin{bmatrix} x & & & + & z & = & 2 \\ & & y & + & z & = & 0 \\ & & & & 0 & = & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We are now in reduced echelon form, with basic variables x and y, and z a free variable. For any choice of z, setting $x = 2 - z$ and $y = -z$ yields a solution to this system of equations. This explains the *free variable* terminology; the values of free variables can be chosen arbitrarily, and then the resulting basic variables can be (uniquely) solved for to yield a solution to the system of equations.

A key takeaway from this example is that placing a matrix in reduced row echelon form does not imply that it has a unique solution! However, it makes finding the *set* of possible solutions a lot easier, as will be discussed in future notes.

In a later note, we will discuss this situation in more detail.

Example 1.6 (Equations with no solution):

$$\begin{bmatrix} x & + & 4y & + & 2z & = & 2 \\ x & + & 2y & + & 8z & = & 0 \\ x & + & 3y & + & 5z & = & 3 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & 3 & 5 & 3 \end{array}\right]$$

To eliminate x from all but the first equation, subtract row 1 from row 2 and row 3.

$$\begin{bmatrix} x & + & 4y & + & 2z & = & 2 \\ & - & 2y & + & 6z & = & -2 \\ & - & y & + & 3z & = & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & -2 & 6 & -2 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

To make 1 the leading coefficient in row 2, divide row 2 by -2.

$$\begin{bmatrix} x & + & 4y & + & 2z & = & 2 \\ & & y & - & 3z & = & 1 \\ & - & y & + & 3z & = & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

To eliminate y from the final equation, add row 2 to row 3.

$$\begin{bmatrix} x & + & 4y & + & 2z & = & 2 \\ & & y & - & 3z & = & 1 \\ & & & & 0 & = & 2 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Even without continuing to the completion of the Gaussian elimination algorithm, the third equation yields a contradiction, $0 = 2$. No choice of x, y, and z will change the rules of mathematics such that $0 = 2$, so there is no solution to this system of equations. If these were experimentally measured results, this contradiction might indicate that our modeling assumptions are incorrect or that there is noise in our measurements.

In fact, this situation comes up frequently in real experiments, and later in this course we'll investigate techniques for dealing with noisy measurements.

Since we ran into a contradiction, we stopped Gaussian elimination early, so the matrix is not yet in reduced row echelon form. If we had continued to the end, we would have still obtained a contradiction (specifically $0 = 1$). In case a contradiction is reached, you can already determine that no solution exists. So, you can stop there, or proceed to the end of the algorithm. Either way, you'll conclude that no solution exists for your system of equations.

Example 1.7 (Another system with infinite solutions):

$$\begin{bmatrix} x & + & y & + & 3z & = & 2 \\ 2x & + & 2y & + & 7z & = & 6 \\ -x & - & y & - & 2z & = & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 2 & 7 & 6 \\ -1 & -1 & -2 & 0 \end{array}\right]$$

To eliminate x from all but the first equation, subtract 2 times row 1 from row 2 and add row 1 to row 3.

$$\begin{bmatrix} x & + & y & + & 3z & = & 2 \\ & & & & z & = & 2 \\ & & & & z & = & 2 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Canceling x from rows 2 and 3 has also canceled y, so we eliminate the next variable, z, from the last row.

$$\begin{bmatrix} x & + & y & + & 3z & = & 2 \\ & & & & z & = & 2 \\ & & & & 0 & = & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, we subtract 3 times row 2 from row 1 to eliminate z from the first equation.

$$\begin{bmatrix} x & + & y & & & = & -4 \\ & & & & z & = & 2 \\ & & & & 0 & = & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here, x and z are basic variables, and y is a free variable. For any choice of y, setting $x = -(4+y)$ and $z = 2$ will yield a solution.

1.7.5 Tomography Revisited

How does what we have learned so far relate back to our tomography example, way back at the start of this note? We know that because our grocer's measurements come from a specific box with a particular assortment of milk, juice, and empty bottles, there must be one underlying solution, but insufficient measurements could give us a system of equations with an infinite number of solutions. So, how many measurements do we need?

Initially, we thought about shining a light vertically and horizontally through the box, giving six total equations because there are three rows and three columns per box. However, there are nine bottles to identify, and therefore nine variables, so we will need nine equations. Based on what you have learned about Gaussian elimination, you now understand that we need at least three more measurements — likely taken diagonally — in order to properly identify the bottles. In coming notes, we will discuss in further detail how you can tell whether or not the nine measurements you choose will allow you to find the solution.