EECS 16A
Module 3
Lecture 4 (GC)
8/5/2014

Topics
Noisy systems
Least squares

- Extra OH: Today 3-4 pm
- HW6B is up

Noisy system: $A\bar{x} = \bar{b} \rightarrow$ no solutions

$A\bar{x} \simeq \bar{b} \Rightarrow A\bar{x} + \bar{e} = \bar{b} \rightarrow$ unique solⁿ $\hat{x}$

$\Rightarrow A\hat{x} + \bar{e} = \bar{b}$

$\Rightarrow A\hat{x} = \bar{b} - \bar{e}$

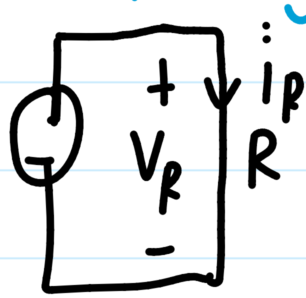
$\Rightarrow A\hat{x} = \bar{b}_0 \quad [\bar{b}_0 = \bar{b} - \bar{e}]$ — (1)

Find $\bar{b}_0$ such than:

1) $A\hat{x} = \bar{b}_0$ has a unique $sol^n = \hat{x}$
i.e. $\bar{b}_0 \in C(A)$

ii) Error between $\bar{b}$ & $\bar{b}_0$ would be minimized: $\bar{e} = \bar{b} - \bar{b}_0$

Ex: Finding R



$\dot{i}_R R = \bar{v}_R$, $\bar{v}_R = \begin{bmatrix}4\\2\end{bmatrix}$, $\bar{\dot{i}}_R = \begin{bmatrix}2m\\0.9m\end{bmatrix}$

$A\bar{x} = \bar{b}$

$\bar{v}_R \to \bar{b}$

$\dot{i}_R \to A$, where $A = [\dot{\bar{a}}_1]$

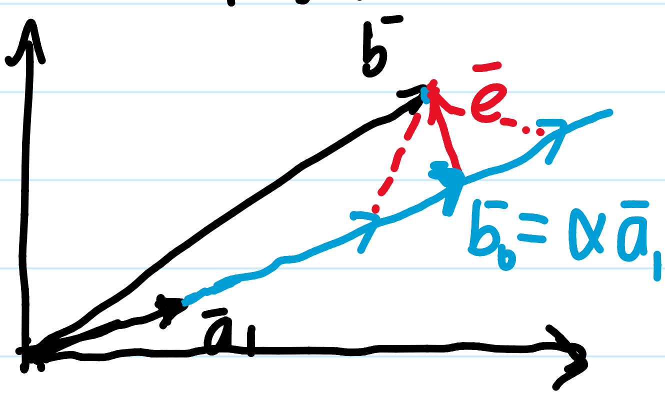
$R \to \bar{x}$

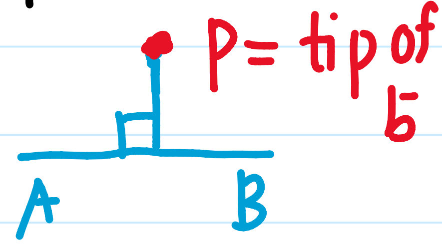
Find approximate sol^n $\hat{\bar{x}} = \hat{R}$

Columnspace of $A = C(A) = \text{span}\left\{\begin{bmatrix}2\\0.9\end{bmatrix}\right\}$

$= \text{span}\{\bar{a}_1\}$

$\bar{b} = \begin{bmatrix}4\\2\end{bmatrix} \notin C(A)$





$\bar{b} = \bar{b}_0 + \bar{e}$

$\bar{b}_0$ is the projection of $\bar{b}$ on $\bar{a}_1$

$\Rightarrow \bar{b}_0 = \text{proj}_{\bar{a}_1} \bar{b}$

$\bar{b}_0 = \text{span}\{\bar{a}_1\} = \alpha \bar{a}_1$

(i) $\Rightarrow$ $\bar{b}_0 = A\hat{x}$

$= [\bar{a}_1]\hat{x} = \hat{x}\bar{a}_1$

Find: $\alpha = \hat{x}$:

$\bar{a}_1$ & $\bar{e}$ are orthogonal

$\Rightarrow \langle \bar{a}_1, \bar{e} \rangle = 0$

$\Rightarrow \langle \bar{a}_1, \bar{b} - \bar{b}_0 \rangle = 0$

★ Properties Dis 6A

$\Rightarrow \underbrace{\langle \bar{a}_1, \bar{b} \rangle - \langle \bar{a}_1, \bar{b}_0 \rangle}_{*} = 0$

$\Rightarrow \langle \bar{a}_1, \bar{b} \rangle - \langle \bar{a}_1, \alpha \bar{a}_1 \rangle = 0 \quad | \; \bar{b}_0 = \alpha \bar{a}_1$

$\Rightarrow \langle \bar{a}_1, \bar{b} \rangle - \underbrace{\alpha \langle \bar{a}_1, \bar{a}_1 \rangle}_{*} = 0$

$\Rightarrow \langle \bar{a}_1, \bar{b} \rangle - \alpha \|a_1\|^2 = 0$

$$\Rightarrow \boxed{\alpha = \hat{x} = \frac{\langle \bar{a}, \bar{b} \rangle}{\|\bar{a}_1\|^2}}$$

$$\hat{R} = \alpha = \frac{\langle \bar{i}_R, \bar{v}_R \rangle}{\|i_R\|^2} = 2.04\,k\Omega$$

What is the projection of $\bar{b}$ on $\text{span}\{\bar{a}_1\}$

$$\boxed{\bar{b}_0 = \alpha \bar{a}_1 = \frac{\langle \bar{a}_1, \bar{b} \rangle}{\|\bar{a}_1\|^2} \bar{a}_1}$$

Noisy measurements: Two variables

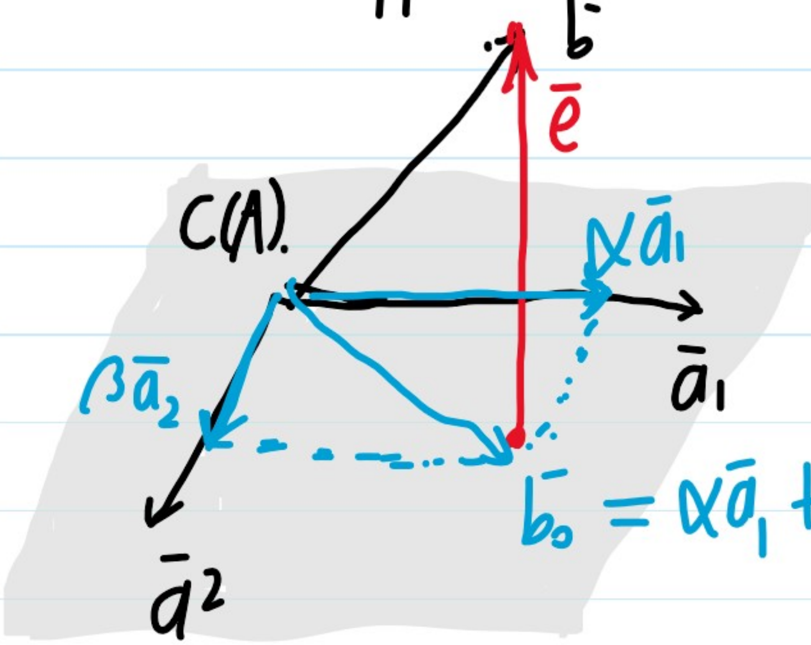
$A\bar{x} \approx \bar{b}$ Noisy

$A\hat{x} = \bar{b} - \bar{e} = \bar{b}_0$ unique solution

$A = \begin{bmatrix} | & | \\ \bar{a}_1 & \bar{a}_2 \\ | & | \end{bmatrix}$ $\quad C(A) = \text{span}\{\bar{a}_1, \bar{a}_2\}$ ↳ plane

$\bar{b} \notin C(A)$ $\quad \bar{b}_0 \in C(A)$

Find approx sol^n $\hat{x}$:



$\bar{b}_0$ is the projection of $\bar{b}$ on $C(A)$:

$\bar{b}_0 = \text{proj}_{C(A)} \bar{b}$

$\bar{b}_0 = A\hat{x} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$ $\quad \Big| \quad \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$

$\Rightarrow \quad \bar{b}_0 = \hat{x}_1 \bar{a}_1 + \hat{x}_2 \bar{a}_2$

$\bar{b}_0 = \alpha \bar{a}_1 + \beta \bar{a}_2$ compare

- Comparing two equations

$\hat{x}_1 = \alpha$ $\quad \hat{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ $\quad$ Graphical sol^n

$\hat{x}_2 = \beta.$

Mathematically:

$\bar{e}$ is orthogonal to $C(A)$

$\bar{e}$ is orthogonal to $\bar{a}_1 \rightarrow \langle \bar{a}_1, \bar{e} \rangle = 0$

$\bar{e}$ is orthogonal to $\bar{a}_2 \rightarrow \langle \bar{a}_2, \bar{e} \rangle = 0$

$$A = \begin{bmatrix} | & | \\ \bar{a}_1 & \bar{a}_2 \\ | & | \end{bmatrix} \quad A^T = \begin{bmatrix} - \bar{a}_1^T - \\ - \bar{a}_2^T - \end{bmatrix}$$

$$A^T \bar{e} = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \bar{e} = \begin{bmatrix} \bar{a}_1^T \bar{e} \\ \bar{a}_2^T e \end{bmatrix} = \begin{bmatrix} \langle \bar{a}_1, \bar{e} \rangle \\ \langle \bar{a}_2, e \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A^T(\bar{b} - \bar{b}_0) = \bar{0}$$

$$\Rightarrow A^T(\bar{b} - A\hat{x}) = \bar{0} \qquad \left| \; \bar{b}_0 = A\hat{x} \right.$$

From $\text{eq}^n(1)$

$$\Rightarrow A^T\bar{b} - A^TA\hat{x} = \bar{0}$$

$$\Rightarrow A^TA\underbrace{\hat{x}} = A^T\bar{b}$$

$$\Rightarrow \underbrace{(A^TA)^{-1}A^TA}_{I}\hat{x} = (A^TA)^{-1}A^T\bar{b}$$

$$\Rightarrow \boxed{\hat{x} = (A^TA)^{-1}A^T \bar{b}}$$ Least squares solution

* Is A^TA invertible?
Proof on Lec 6P

A^TA is square matrix

$\hat{x}$ was derived by minimizing $\|\bar{e}\|$

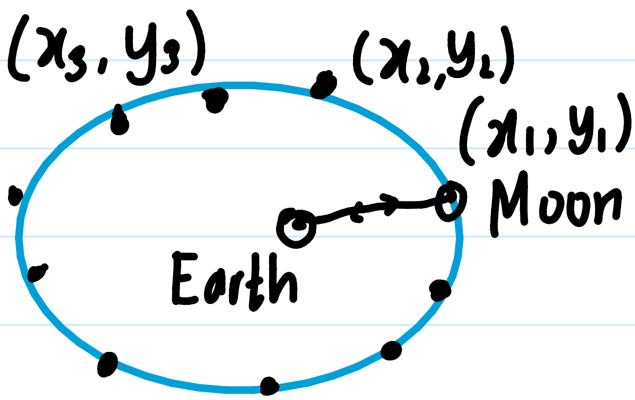
$$\|e\| = \sqrt{\underbrace{e_1^2 + e_2^2}_{\text{squares of errors}} \ldots}$$

→ minimized
"Least Squares"

$$\boxed{\begin{aligned} \bar{b}_0 &= \text{proj}_{C(A)}\bar{a} = A\hat{x} \\ &= A(A^TA)^{-1}A^T\bar{b} \end{aligned}}$$

Polynomial Fitting:

Orbit of the moon : ellipse



x	y
x_1	y_1
x_2	y_2
x_3	y_3
$\vdots$	$\vdots$

General equation of an ellipse :

$$ax^2 + by^2 + cxy + dx + ey = 1$$

Known: x, y

Unknown: a, b, c, d, e

For (x_1, y_1): $ax_1^2 + by_1^2 + cx_1y_1 + dx_1 + ey_1 = 1$

For (x_2, y_3): $ax_2^2 + by_2^2 \dots = 1$

$\vdots$

$ax_n^2 + by_n^2 + \dots = 1$

Matrix-vector form:

$$\underbrace{\begin{bmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 \\ x_2^2 & y_2^2 & \cdots & & \cdots \\ & & & & \\ x_n^2 & \cdots & & & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}}_{\bar{x}_{ellipse}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}}_{\bar{b}}$$

$$\bar{x}_{ellipse} = (A^T A)^{-1} A \bar{b} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

We need an overdetermined system,
i.e. # measurements > # variables
Makes the system less susceptible to noise.

Quadratic eq^n fitting: Dis 6D

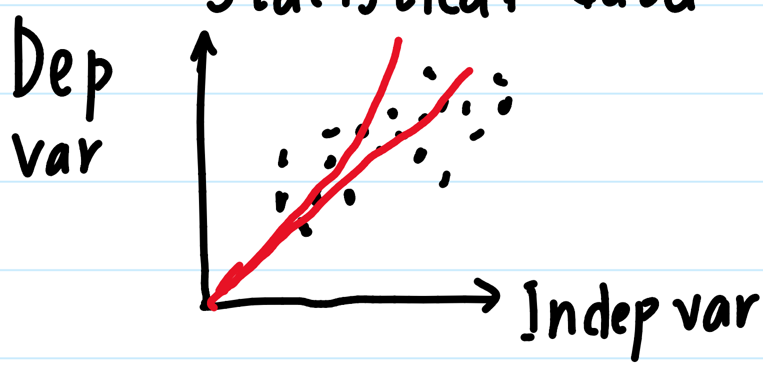
$$y = ax^2 + bx + c$$

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x & 1 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

x	y
x_1	y_1
x_2	y_2

Regression analysis:

- Statistical data



- Fit a curve using least squares
- Modeling
- Prediction via extrapolation

Smaller $\|e\|^2 \rightarrow$ better fit

$y_1 = ax + b \quad : \|e_1\|^2$

$y_2 = ax^2 + bx + c \quad : \|e_2\|^2$

compare numerical values

iPython demo:

predict global temperature anomaly for 2019, using

- 1850–2009 data of CO_2 emission
- Temperature anomaly vs. CO_2 emission