EECS 16A Designing Information Devices and Systems I
Summer 2020 Homework 5B

This homework is due Sunday, August 2, 2020, at 23:59.
Self-grades are due Wednesday, August 5, 2020, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

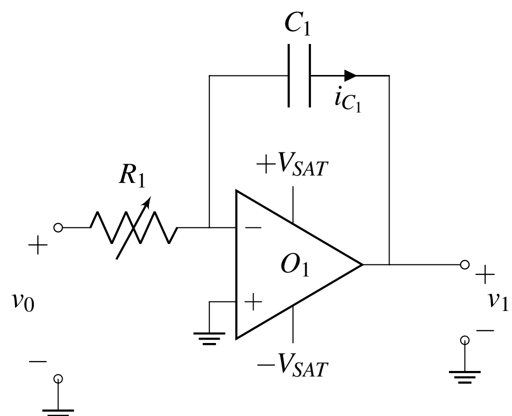
Homework Learning Goals: *The objective of this homework is to present a variety of real-world circuit design problems that can be tackled using Op-Amp and switched capacitor circuits.*

1. PRACTICE: Integration using Op-amps

As we have seen already it is useful in several applications to create triangular voltages (also known as voltage ramps). Remember for instance the HW 8 problem where we built a circuit that could measure the level of water in a tank by "integrating" current on a capacitor whose value changed with the level of water in the tank. In this problem, you will be analyzing a circuit that produces a voltage ramp using a voltage source and an op-amp in negative feedback.

(a) One of the circuit blocks you can use to generate the triangular waveform is the integrator. An integrator outputs the integral of the input signal. For the circuit given below express v_1 in terms of R_1, C_1, v_0, and t, assuming v_0 is not varying with time. What is the slope of this voltage ramp? You may also assume that capacitor C_1 has $0V$ across it at time $t = 0$.

Hint: You will have to apply KCL, and use the fact that the current flowing through a capacitor is given by $I = C\frac{dV}{dt}$.



Solution:

Let's write the KCL equation at V^- assuming all currents are leaving that node.

$$\begin{aligned}
i_{R_1} + i_{C_1} &= 0 \\
i_{R_1} &= -i_{C_1} \\
i_{C_1} &= C_1 \frac{d(0 - v_1(t))}{dt} \\
\frac{0 - v_0}{R_1} &= C_1 \frac{d(v_1(t) - 0)}{dt} \\
-\frac{v_0}{R_1 C_1} &= \frac{dv_1(t)}{dt} \\
v_1(t) &= -\frac{1}{R_1 C_1} \int_0^t v_0 d\tau \\
v_1(t) &= -\frac{1}{R_1 C_1} v_0 t
\end{aligned}$$

Which means that the **slope** of the ramp is going to be equal to $-\frac{1}{R_1 C_1} v_0$

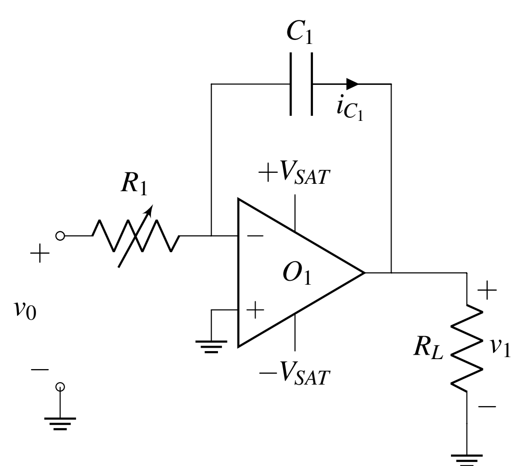
(b) What is the value of the current i_{C_1} flowing through capacitor C_1? How does the capacitor current change if we double C_1? How does the slope of the ramp change if we double capacitor C_1?
Note: the current direction is specified in the figure above.

Solution: As calculated in part (a),

$$i_{C_1} = -i_{R_1} = -\frac{0 - v_0}{R_1} = \frac{v_0}{R_1}$$

Since this value is **independent** of C_1 the current i_{C_1} will **not** change if we double the capacitor. On the other hand, the **slope** of the ramp is equal to $-\frac{1}{R_1 C_1} v_0$ so we will get half the slope if we double the capacitor. Intuitively, this makes sense since we would expect charging a capacitor twice as large with the same current to take twice as much time.

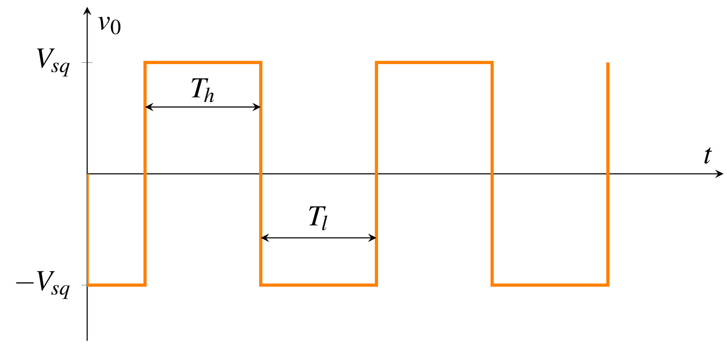
(c) If we connect a load resistance at the output of the circuit, as shown in the figure below, does the output voltage v_1 change, from what you calculated in part (a)? Why or why not?



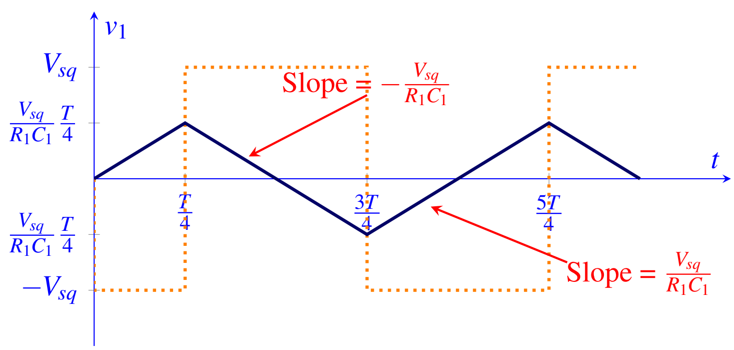
Solution:

The output voltage will not change if we connect a load at the output since it is solely defined by the current flowing through capacitor C_1. This current as we have already seen in part (b) is dependent only upon the input voltage v_0 and the resistor R_1. Any extra curent through the load resistor will be provided by the output of the op-amp, which is within the capabilities of an ideal op-amp.

(d) If v_0 varies with time as shown in the following diagram, plot v_1 for $t = 0$ to $t = 1.5T$, where $T = 2T_h = 2T_l$. In your plot indicate an algebraic expression for the slope (as a function of R_1, C_1 and v_{sq}) and add tick marks on the x and y axis indicating the time and voltage values where the ramp slope changes. You may assume again that capacitor C_1 has 0V across it at time $t = 0$.



Solution:



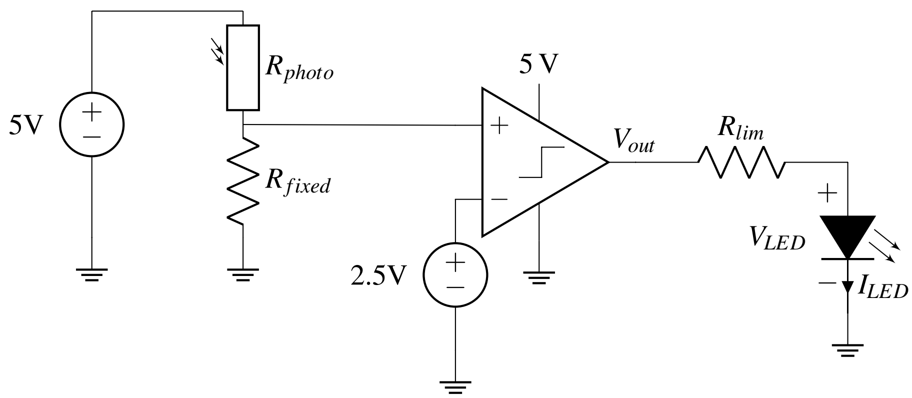
(e) **Practice (Optional):** Prove that the units of RC in SI are seconds.

Solution: We know that R is measured in Ohms and C in Farads so we can write that RC is measured in:

$$\Omega\text{F} = \frac{\text{V}}{\text{A}}\frac{\text{C}}{\text{V}} = \frac{\text{C}}{\text{A}} = \frac{\text{C}}{\text{C}/\text{sec}} = \text{sec}$$

2. PRACTICE: LED Alarm Circuit

One day, you come back to your dorm to find that your favorite candy has been stolen. Determined to catch the perpetrator red-handed, you decide to put the candy inside a kitchen drawer. Using the following circuit design, you would like to turn on a light-emitting diode (LED) “alarm” if the kitchen drawer is opened.



Note R_{photo} is a photoresistor, which acts like a typical resistor but changes resistance based on the amount of light it is exposed to. This photoresistor is located inside the kitchen drawer, so we can tell when the drawer is opened or closed.

V_{LED} indicates the voltage across the LED; we will guide you through the IV behavior of this element later in the problem. The LED is located in your room (and connected to a long wire going to the kitchen), so that you can remotely tell when the kitchen drawer has been opened.

(a) What is V_+, the voltage at the positive voltage input of the comparator? Your answer should be written in terms of R_{photo} and R_{fixed}.

Solution: V_+ is the output of a voltage divider:

$$V_+ = \frac{R_{fixed}}{R_{fixed} + R_{photo}} \cdot 5\,\text{V}$$

(b) We now want to choose a value for R_{fixed}. From the photoresistor's datasheet, we see the resistance in "light" conditions (i.e. drawer open) is $1\,\text{k}\Omega$. In "dark" conditions (i.e. drawer closed), the resistance is $10\,\text{k}\Omega$.

To ensure the comparator detects the light condition with more tolerance, we decide to design R_{fixed} so that V_+ is $3\,\text{V}$ under the "light" condition. Solve for the value of R_{fixed} to meet this specification.

Solution: We start from the voltage divider equation we derived in the previous part:

$$V_+ = \frac{R_{fixed}}{R_{fixed} + R_{photo}} \cdot 5\,\text{V}$$

Now we plug in the known values, $V_+ = 3\,\text{V}$ and $R_{photo} = 1\,\text{k}\Omega$.

$$3\,\text{V} = \frac{R_{fixed}}{R_{fixed} + 1000\Omega} \cdot 5\,\text{V}$$

Solving this equation, we get $R_{fixed} = 1.5\,\text{k}\Omega$.

(c) Write down V_{out} with any conditions in terms of V_+. For simplicity, consider the case when $V_+ \neq V_-$ and assume the comparator is ideal.

Solution:

Since the comparator is ideal, we know that V_{out} will be the voltage at either the positive rail ($5\,\text{V}$) or at the negative rail ($0\,\text{V}$) when $V_+ \neq V_-$. Which voltage depends on if V_+ is greater than V_- or not. Since V_- is $2.5\,\text{V}$, we get the following piecewise equation for V_{out}:

$$V_{out} = \begin{cases} 5\,\text{V}, & V_+ > 2.5\,\text{V} \\ 0\,\text{V}, & V_+ < 2.5\,\text{V} \end{cases}$$

(d) Using your answers to the previous parts, write down V_{out} with the conditions on its output in terms of R_{photo}. You can substitute the value of R_{fixed} you found in part (b). As before, you can assume that $V_+ \neq V_-$ and the comparator is ideal.

Solution:

We substitute the equations for V_+ into the equation for V_{out}:

$$V_{out} = \begin{cases} 5\,\text{V}, & \frac{R_{fixed}}{R_{fixed}+R_{photo}} \cdot 5\,\text{V} > 2.5\,\text{V} \\ 0\,\text{V}, & \frac{R_{fixed}}{R_{fixed}+R_{photo}} \cdot 5\,\text{V} < 2.5\,\text{V} \end{cases}$$

Plugging in $R_{fixed} = 1.5\,\text{k}\Omega$ from part (b), we can get the following in terms of R_{photo}:

$$V_{out} = \begin{cases} 5\,\text{V}, & R_{photo} < 1.5\,\text{k}\Omega \\ 0\,\text{V}, & R_{photo} > 1.5\,\text{k}\Omega \end{cases}$$

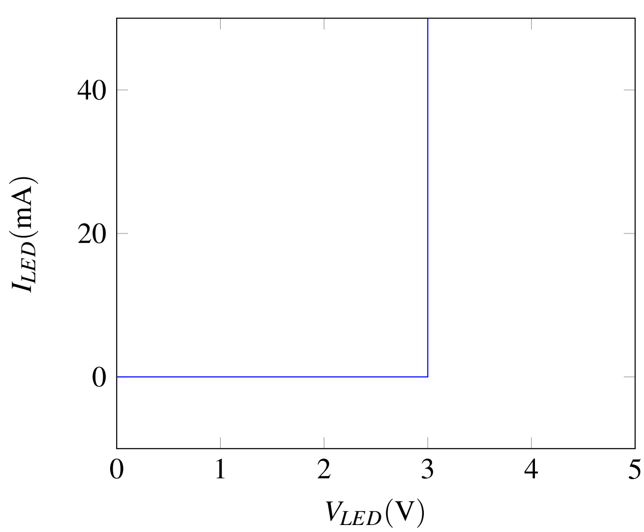
(e) From the design steps in the previous parts, we have designed a circuit that outputs non-zero voltage when the photoresistor is exposed to light (i.e. kitchen drawer open). We now want to design the LED portion of the circuit, so we get a visual alarm when the drawer is open.

From the LED's datasheet, the forward voltage, V_F is $3\,\text{V}$. Essentially, if V_{LED} is less than this voltage, the LED won't light up and I_{LED} will be $0\,\text{A}$.

Here is an idealized IV curve of this LED. The LED behaves in one of the following two modes:

i. If the voltage across the LED is less than $V_F = 3\,\text{V}$ or if $I_{LED} < 0\,\text{A}$, then the LED acts like an open circuit.

ii. If the voltage across the LED is $V_F = 3\,\text{V}$, then the LED acts like a voltage source, except that it only allows positive current flow (i.e. only in the direction of current marked on the circuit diagram).



To avoid exceeding the power rating of the LED (and having it burn out), the recommended value for I_{LED} is $20\,\text{mA}$.

Find the value of the current-limiting resistor, R_{lim}, such that when the photoresistor is in the "light" condition, $I_{LED} = 20\,\text{mA}$.

Solution: When the photoresistor is in the "light" condition, $R_{photo} = 1\,\text{k}\Omega$ so that $V_{out} = 5\,\text{V}$ per the analysis in the previous part. This implies that $V_{LED} = V_F$ and the LED acts like a power supply with positive current flow when in the "light" condition.

Using Ohm's Law and noting that the same current passes through R_{lim} and the LED itself,

$$V_{out} - V_F = I_{LED} R_{lim}$$

Rearranging and plugging in values when in the "light" condition:

$$R_{lim} = \frac{V_{out} - V_F}{I_{LED}}$$

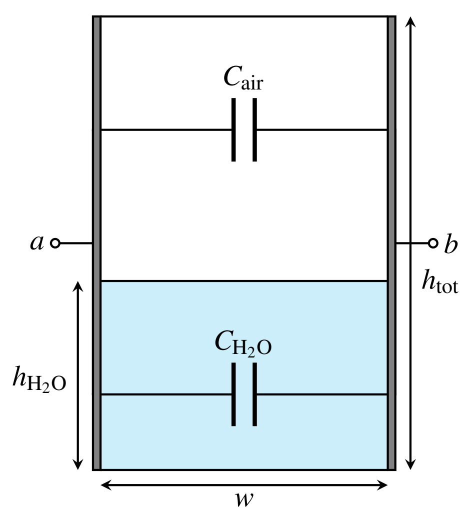
$$R_{lim} = \frac{5 - 3\text{ V}}{0.02\text{ A}}$$

$$R_{lim} = 100\,\Omega$$

Note that when $V_{out} < 3\,\text{V}$, the LED will not light up and I_{LED} will be $0\,\text{mA}$. Thus by our design of the voltage divider, we were able to ensure the LED lights up only if the drawer is opened.

3. PRACTICE: Rain Sensor v2.0

In a previous homework, we analyzed a rain sensor built by a lettuce farmer in Salinas Valley. They used a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside. The width and length of the tank are both w (i.e. the base is square), and the height of the tank is h_{tot}.



As your EE16A circuits toolkit is now complete with capacitors, op-amps, and switches, we will revisit this problem to improve the readout electronics. The goal is to create a circuit block that will output voltage as a linear function of the water height, $h_{\text{H}_2\text{O}}$.

(a) What is the capacitance between terminals a and b when the tank is empty, C_{empty}? Again, the height of the water in the tank is $h_{\text{H}_2\text{O}}$. Modeling the tank as a pair of capacitors in parallel, find the total capacitance C_{tank} between the two plates. Can you write C_{tank} as a function of C_{empty}?

Note: The permittivity of air is ε, and the permittivity of rainwater is 81ε.

Solution:

$$C_{\text{empty}} = \frac{\varepsilon_{\text{air}} h_{\text{tot}} w}{w} = \varepsilon h_{\text{tot}}$$

For C_{tank}, we can break the total capacitance into two parts. First, let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\varepsilon_{\text{H}_2\text{O}} h_{\text{H}_2\text{O}} w}{w} = 81\varepsilon h_{\text{H}_2\text{O}}$$

And now, we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\varepsilon_{\text{air}} (h_{\text{tot}} - h_{\text{H}_2\text{O}}) w}{w} = \varepsilon (h_{\text{tot}} - h_{\text{H}_2\text{O}})$$

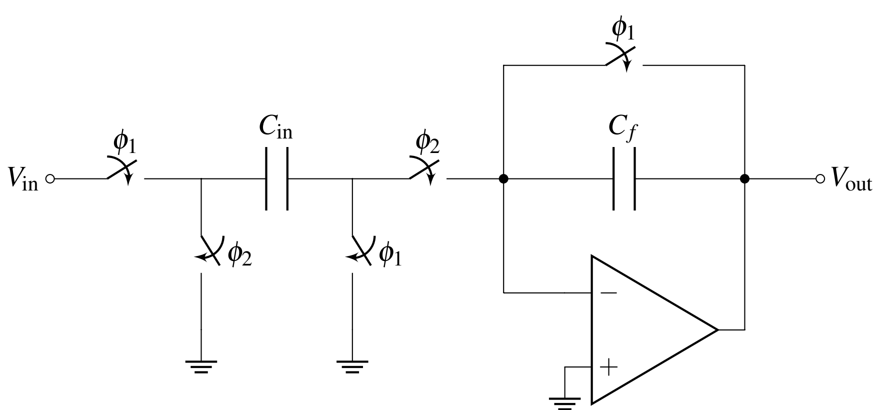
Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

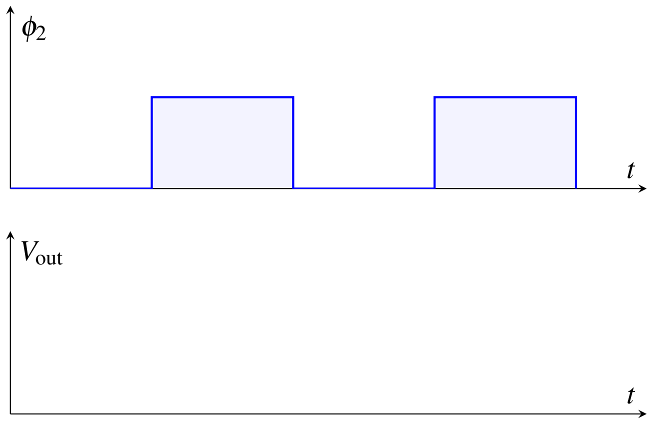
$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \varepsilon (h_{\text{tot}} + 80 h_{\text{H}_2\text{O}})$$

Now, we can rewrite the above equation to show the dependence on C_{empty}:

$$C_{\text{tank}} = C_{\text{empty}} + 80\varepsilon h_{\text{H}_2\text{O}}$$

(b) Here, we will analyze a circuit that transfers all charges for efficient readout. For the circuit below, draw the output waveform of v_{out} as a function of v_{in}, C_f, and C_{in}.





Solution:

In ϕ_1, C_{in} gets charged up to

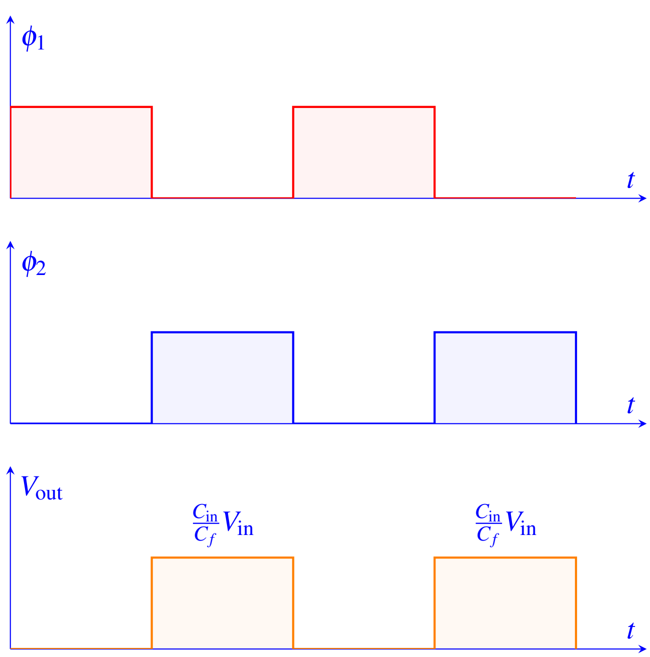
$$Q_{C_{in},1} = C_{in} v_{in}.$$

On the other hand, since v_{out} and v_- are at zero potential, no charge is stored on C_f, so

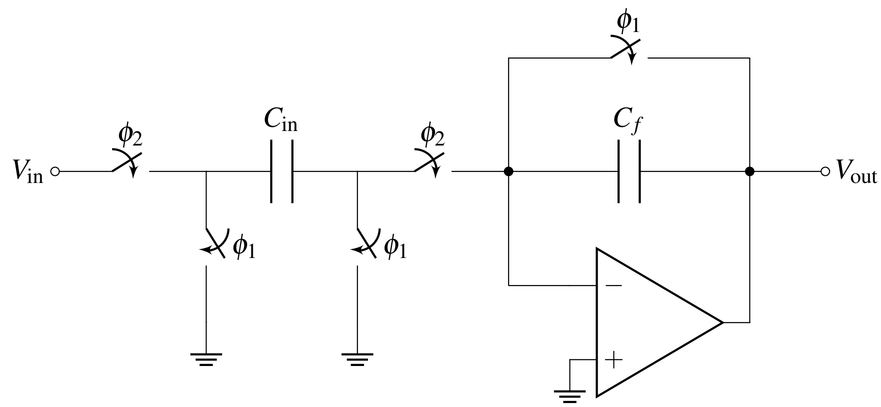
$$Q_{C_f,1} = 0.$$

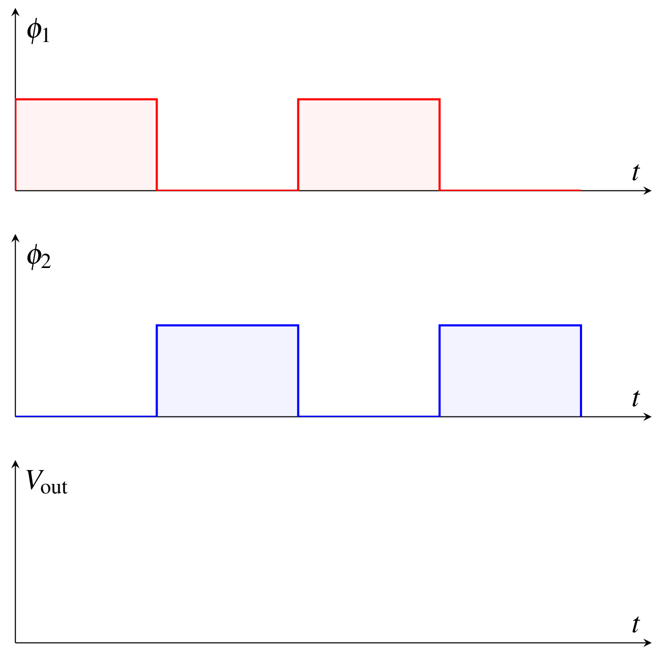
In ϕ_2, the positive plate of C_{in} is shorted to ground, which forces all of the negative charges to move to the left plate of C_f. Now using charge conservation,

$$Q_{C_{in},1} + Q_{C_f,1} = Q_{C_{in},2} + Q_{C_f,2}$$
$$C_{in} v_{in} + 0 = 0 + C_f v_{out}$$
$$v_{out} = \frac{C_{in}}{C_f} v_{in}$$



(c) We examined the non-inverting configuration in the previous part- now, we will look into the inverting configuration. For the circuit below, draw the output waveform of v_{out} as a function of v_{in}, C_f, and C_{in}.





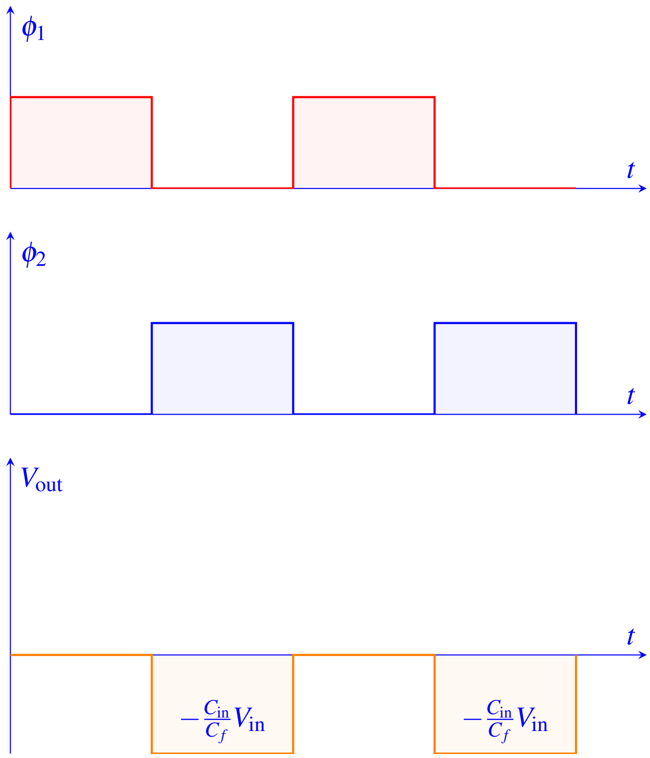
Solution:

In ϕ_1, the voltage across both the capacitors is zero, so

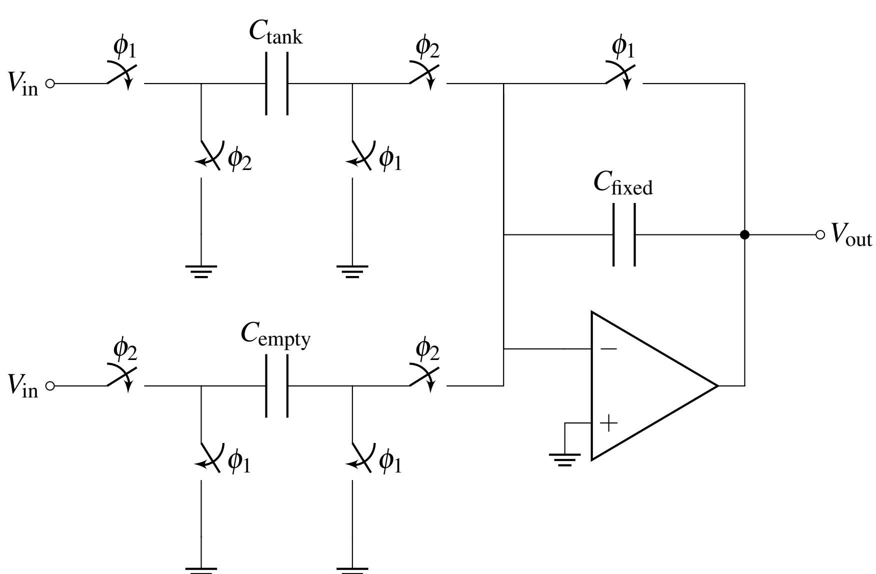
$$Q_{C_{\text{in}},1} = Q_{C_f,1} = 0.$$

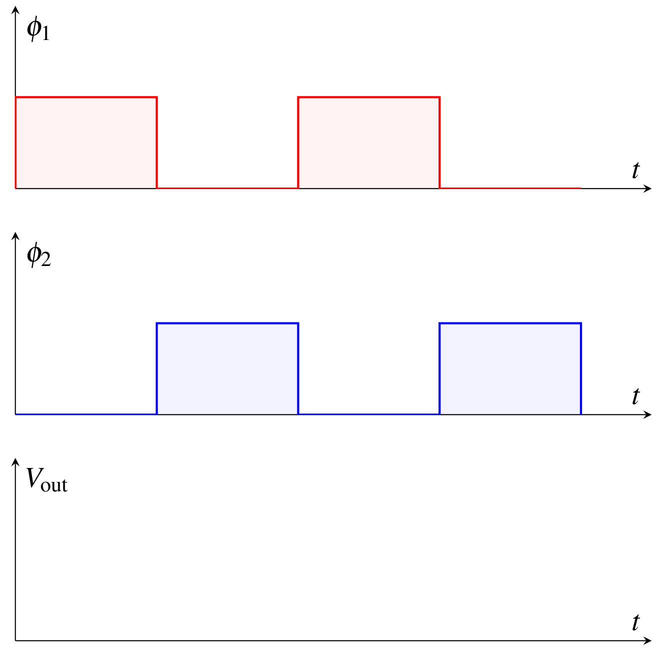
In ϕ_2, C_{in} gets charged up to $Q_{C_{\text{in}},2} = C_{\text{in}}v_{\text{in}}$. Since C_f is in series, C_f stores the same amount of charge but with opposite polarity, so $Q_{C_f,2} = -C_f v_{\text{out}}$.

$$\begin{aligned} Q_{C_{\text{in}},2} &= Q_{C_f,2} \\ C_{\text{in}}v_{\text{in}} &= -C_f v_{\text{out}} \\ v_{\text{out}} &= -\frac{C_{\text{in}}}{C_f}v_{\text{in}} \end{aligned}$$



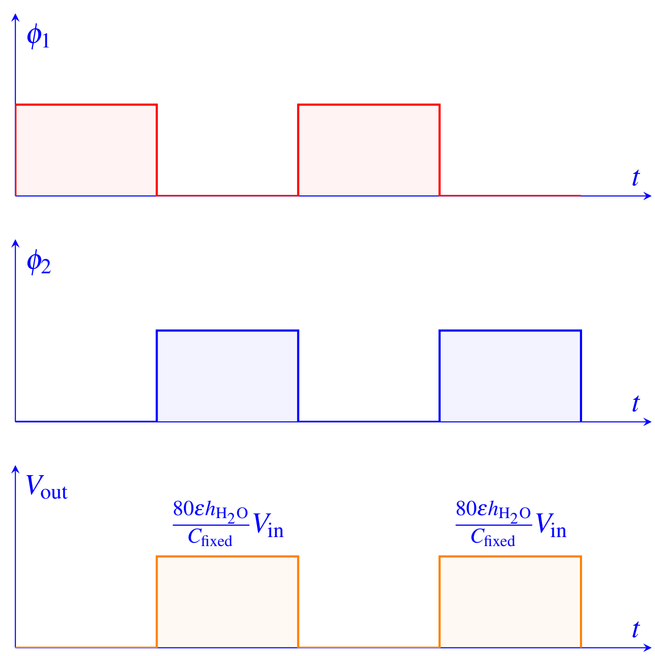
(d) With the help of the basic circuit blocks shown in parts (b) and (c), we will now implement a circuit that will output voltage as a linear function of the water height, h_{H_2O}. In addition to the rain-sensing capacitor, we will use two fixed value capacitors C_{fixed} and C_{empty}. Use the values obtained in part (a) for C_{tank} and C_{empty}. For the circuit below, draw the output waveform of v_{out} as a function of v_{in}, C_{fixed}, ε, and h_{H_2O}.





Solution:

$$\begin{aligned} v_{out} &= \left(\frac{C_{tank}}{C_{fixed}} - \frac{C_{empty}}{C_{fixed}} \right) v_{in} \\ &= \frac{C_{empty} + 80\varepsilon h_{H_2O} - C_{empty}}{C_{fixed}} v_{in} \\ &= \frac{80\varepsilon h_{H_2O}}{C_{fixed}} v_{in} \end{aligned}$$



4. PRACTICE: Island Karaoke Machine

You're stuck on a desert island and everyone is bored out of their minds. Fortunately, you have your EE16A lab kit with op-amps, wires, resistors, and your handy breadboard. You decide to build a karaoke machine. You recover one speaker from the crash remains and use your iPhone as your source. You know that many songs put instruments on either the "left" or the "right" channel, but the vocals are usually present on both channels with equal strength.

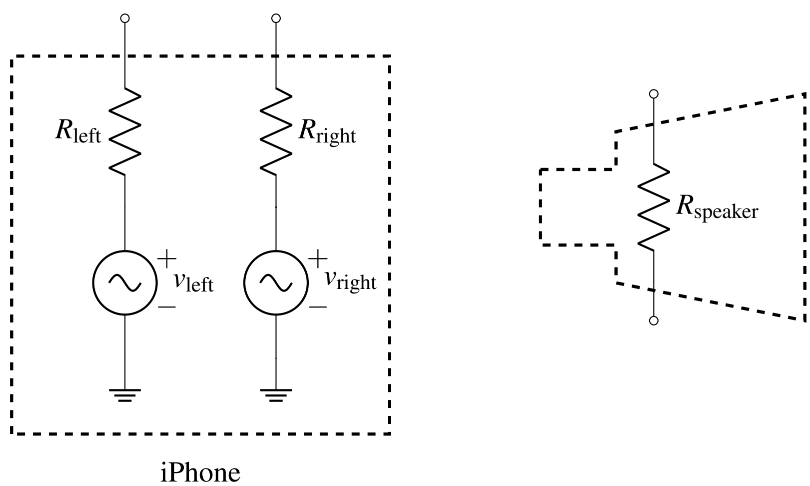
The Thevenin equivalent model of the iPhone audio jack and speakers is shown below. We assume that the audio signals v_{left} and v_{right} have equivalent source resistance of the left/right audio channels of $R_{\text{left}} = R_{\text{right}} = 3\,\Omega$. The speaker has an equivalent resistance of $R_{\text{speaker}} = 4\,\Omega$.

For this problem, we'll assume that the vocals are present on both left and right channels, but the instruments are only present on the right channel, i.e.

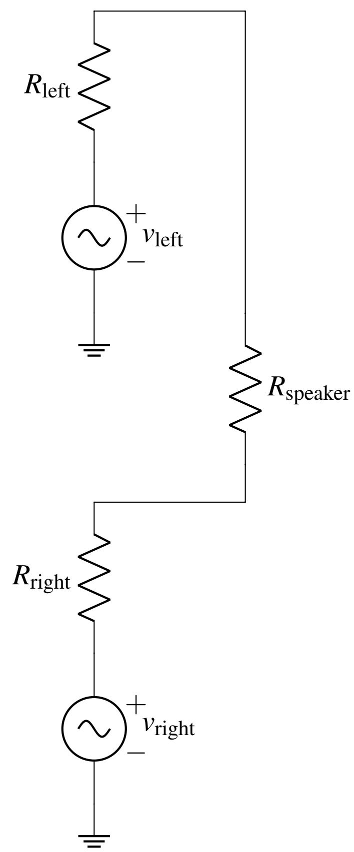
$$v_{\text{left}} = v_{\text{vocals}}$$
$$v_{\text{right}} = v_{\text{vocals}} + v_{\text{instrument}},$$

where the voltage source v_{vocals} can have values anywhere in the range of $\pm 120\,\text{mV}$ and $v_{\text{instrument}}$ can have values anywhere in the range of $\pm 50\,\text{mV}$.

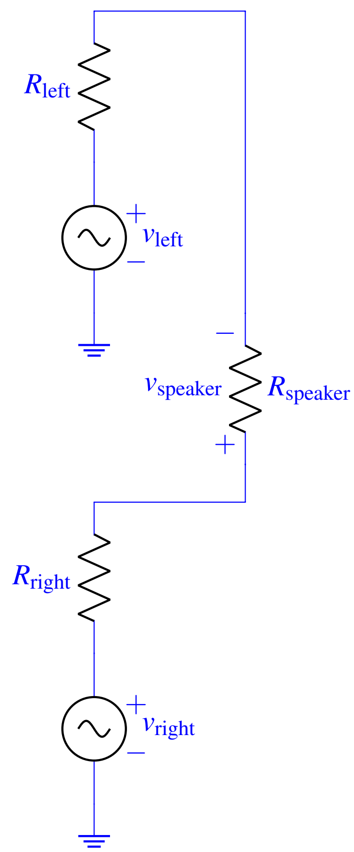
What is the goal of a karaoke machine? The ultimate goal is to *remove* the vocals from the audio output. We're going to do this by first building a circuit that takes the left and right audio outputs of the smartphone and then calculates its difference. Let's see what happens.



(a) One of your island survivors suggests the following circuit to do this. Calculate the voltage across the speaker as a function of v_{vocals} and $v_{\text{instruments}}$.
Does the voltage across the speaker depend on v_{vocals}? What do you think the islanders will hear – vocals, instruments, or both?



Solution:
Let's mark the voltage across the speaker, v_{speaker}, from bottom to top as in the figure:



We can apply the principle of superposition to solve for v_{speaker}. First, we solve for the voltage across the speaker when only v_{left} is on. Let's call this $v_{\text{speaker,left}}$. Notice that the circuit becomes a voltage divider. Therefore, we get

$$-v_{\text{speaker,left}} = \frac{v_{\text{left}} R_{\text{speaker}}}{R_{\text{speaker}} + R_{\text{left}} + R_{\text{right}}} = \frac{4v_{\text{vocals}}}{10} = 0.4v_{\text{vocals}},$$

giving

$$v_{\text{speaker,left}} = -0.4v_{\text{vocals}}.$$

Similarly, we solve for the voltage across the speaker when only v_{right} is on. Let's call this $v_{\text{speaker,right}}$. Again, notice that the circuit becomes a voltage divider. Therefore, we get

$$v_{\text{speaker,right}} = \frac{v_{\text{right}} R_{\text{speaker}}}{R_{\text{speaker}} + R_{\text{left}} + R_{\text{right}}} = \frac{4(v_{\text{vocals}} + v_{\text{instrument}})}{10} = 0.4(v_{\text{vocals}} + v_{\text{instrument}}).$$

Superposition tells us that $v_{\text{speaker}} = v_{\text{speaker,left}} + v_{\text{speaker,right}} = 0.4v_{\text{instrument}} = 0.4 \cdot 50\,\text{mV} = 20\,\text{mV}$. What did you notice? The vocals got canceled out! The islanders will only hear the instruments, just as they wanted.

(b) We need to boost the sound level to get the party going. We can do this by *amplifying* both v_{left} and v_{right}. Keep in mind that we could use inverting or non-inverting amplifiers.

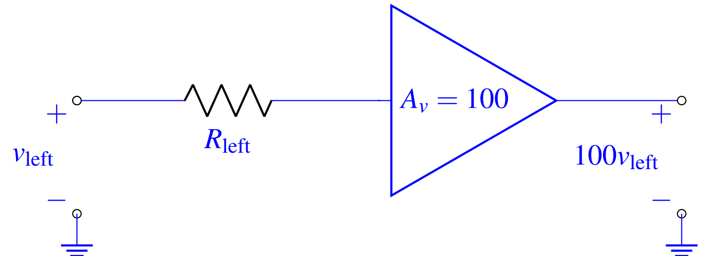
Let's assume, just for this part, that we have already implemented circuits that amplify v_{left} and v_{right} by some factor A_v (Consider $A_v = 100$ for this part). We now have two voltages, v_{Gl} and v_{Gr} that are $A_v \cdot v_{\text{left}}$ and $A_v \cdot v_{\text{right}}$ respectively. Use v_{Gl} and v_{Gr} to get $A_v \cdot v_{\text{instrument}}$ across R_{speaker}.

Solution:

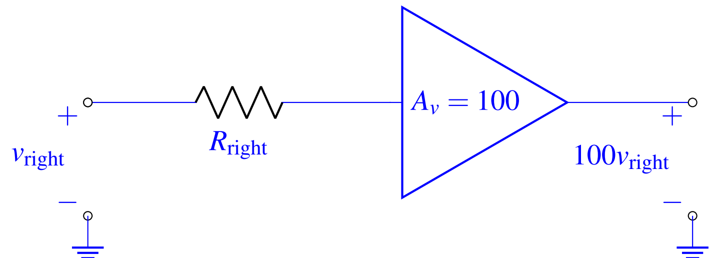
Note: In the following figures, we use a symbolic representation of the amplifier with gain $A_v = 100$. We will add in the corresponding amplifier circuit in part c.

We have three components of the circuit we want to build that we already know:

- The part of the circuit that amplifies v_{left} to $100v_{\text{left}}$, which we can draw as below:

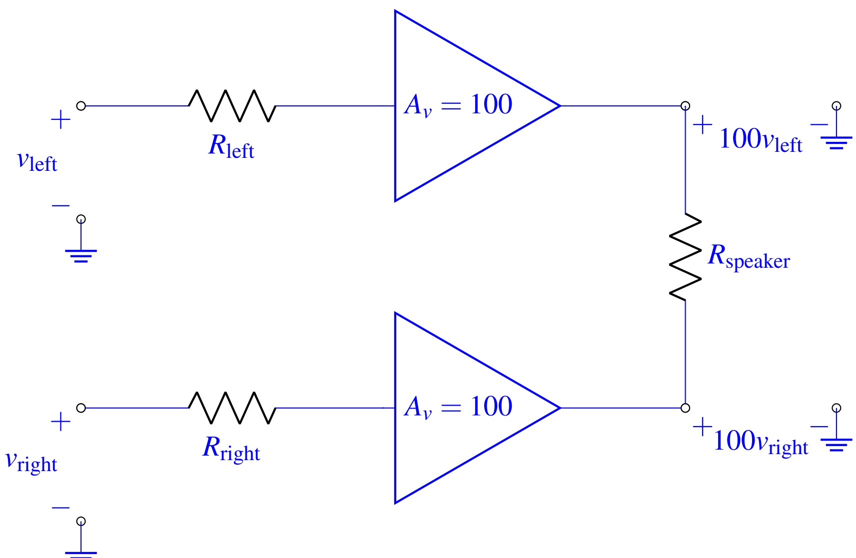


- The part of the circuit that amplifies v_{right} to $100v_{\text{right}}$, which we can draw as below:



- The speaker

If we want to take the difference of the two amplified outputs across the speaker, all we need to do is connect the output terminals of the first two components to the terminals of the speaker as shown below:



You can see this solution taking inspiration from part (a). Why do we get exactly $100(v_{\text{right}} - v_{\text{left}})$ across the speaker? Why does the voltage not divide as before?

To answer this, look back at the circuit for the non-inverting amplifier. If we solve for the Thevenin output resistance of this circuit, we will find that it is zero. Furthermore, the Thevenin voltage will be $100v_{\text{left}}$ (or $100v_{\text{right}}$). This implies that, no matter what R_{left} or R_{right} is, we are going to only see $100v_{\text{left}}$ or $100v_{\text{right}}$ at the output.

(c) Now, you want ± 2 V across the speaker to get the party going. Using the scheme in part (b), design a circuit that takes in v_{left} and v_{right} and outputs an amplified version of $v_{\text{instrument}}$ across the speaker with the range of ± 2 V. You need to design both amplifiers with the right gain A_v to achieve this.

You can use up to two op-amps, and each of them can be inverting or non-inverting.

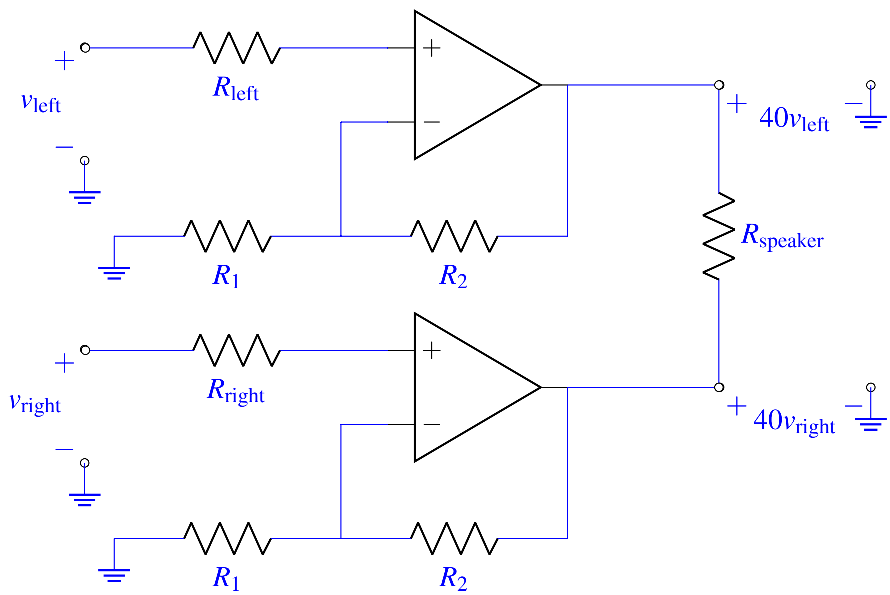
Solution:

Feed the non-ideal voltage source $\{v_{\text{left}}, R_{\text{left}}\}$ into a non-inverting amplifier with gain A_v and the non-ideal voltage $\{v_{\text{right}}, R_{\text{right}}\}$ into another non inverting amplifier with gain A_v. (We have a different gain from the previous part, which we need to determine.) Then connect the two outputs across R_{speaker} as shown in the previous part.

In this circuit, we will get $v_{\text{speaker}} = A_v \cdot v_{\text{instrument}}$. Since $v_{\text{instrument}}$ has a range of ± 50 mV, v_{speaker} will have a range of $\pm 50\,\text{mV} \cdot A_v = \pm 0.05 \cdot A_v$V. Now we need $\pm 0.05 \cdot A_v\text{V} = \pm 2$, i.e. $A_v = 40$.

Therefore, we want to design a non-inverting amplifier with voltage gain of 40.

We can use the circuit schematic from part (b), but now, we just need to design the non-inverting amplifier with op-amps to have gain 40. We get the equivalent circuit below:

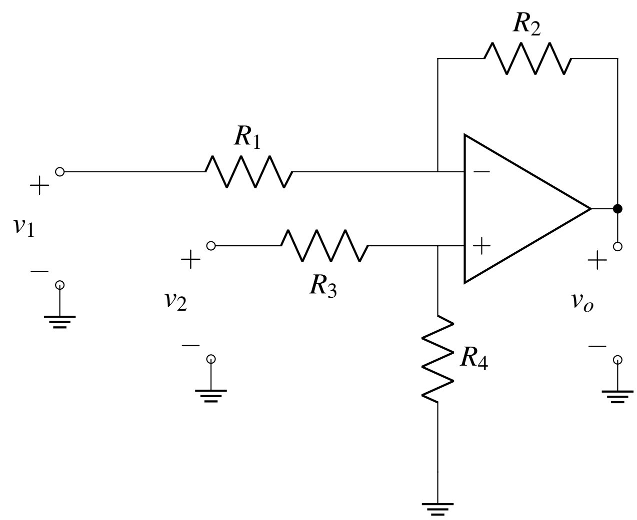


Now, we need to find R_1 and R_2.

$$A_v = 1 + \frac{R_2}{R_1}$$

Therefore, we can then choose any R_1 and R_2 such that $\frac{R_2}{R_1} = 39$. Note that there are multiple ways of choosing them. One such choice is $R_1 = 1\,\text{k}\Omega$ and $R_2 = 39\,\text{k}\Omega$, for instance.

(d) The trouble with the approach in part (c) is that multiple op-amps are required. Let's say you only have one op-amp with you. What would you do? One night in your dreams, you have an inspiration. Why not combine the inverting and non-inverting amplifier into one, as shown below!



If we set $v_2 = 0\,\text{V}$, what is the output v_o in terms of v_1? (This is the inverting path.)

Solution:

If we set $v_2 = 0\,\text{V}$, we would get $u_+ = 0\,\text{V}$. Applying the Golden Rules, we will get $u_- = u_+ = 0\,\text{V}$. Writing KCL at the $-$ terminal of the op-amp, we get

$$\frac{v_1 - 0}{R_1} = \frac{0 - v_{o,1}}{R_2},$$

which gives

$$v_{o,1} = \frac{-v_1 R_2}{R_1}.$$

(e) If we set $v_1 = 0\,\text{V}$, what is the output v_o in terms of v_2? (This is the non-inverting path.)

Solution:

If we set $v_1 = 0\,\text{V}$, we would get $u_+ = \frac{v_2 R_4}{R_3 + R_4} = u_-$. Writing KCL at the $-$ terminal gives

$$\frac{0 - u_-}{R_1} = \frac{u_- - v_{o,2}}{R_2},$$

which gives

$$v_{o,2} = u_- \left(1 + \frac{R_2}{R_1}\right) = v_2 \left(\frac{R_4}{R_3 + R_4}\right)\left(1 + \frac{R_2}{R_1}\right).$$

(f) Now, determine v_o in terms of v_1 and v_2. (*Hint:* Use superposition.) Choose values for R_1, R_2, R_3 and R_4, such that the speaker has $\pm 2\,\text{V}$ across it.

Solution:

By the principle of superposition,

$$v_o = v_{o,1} + v_{o,2}.$$

If we set $v_1 = v_{\text{left}}$ and $v_2 = v_{\text{right}}$, we'd ideally want $v_o = -40v_1 + 40v_2$. We can choose R_1, R_2, R_3 and R_4, so that this happens.

How do we do this? Let's do this in steps. First, note that, looking for the expression for $v_{o,1}$, we'll want $\frac{R_2}{R_1} = 40$. Therefore, we can choose any values of R_2 and R_1, such that this happens. One such choice is $R_1 = 1\,\text{k}\Omega$ and $R_2 = 40\,\text{k}\Omega$. Then, plug that into the expression of $v_{o,2}$, and the condition we now want is

$$\left(\frac{R_4}{R_3+R_4}\right)\left(1+\frac{R_2}{R_1}\right) = 40,$$

which gives us

$$\frac{R_4}{R_3+R_4} = \frac{40}{41}.$$

Thus, we need to choose R_3 and R_4. As before, we can choose these values in many ways. One such choice is $R_4 = 40\,\text{k}\Omega$ and $R_3 = 1\,\text{k}\Omega$.

Note: Keep in mind that, for this problem, we actually assumed that $v_1 = v_{\text{left}}$ and $v_2 = v_{\text{right}}$, which would mean that we are ideally connecting v_{left} and v_{right} as inputs. However, in reality, we're actually connecting the outputs from the iPhone as inputs. This means that R_{left} and R_{right} will also actually affect the output.

With this effect, we will actually get

$$v_o = -\frac{v_1 R_2}{R_{1,eq}} + v_2\left(\frac{R_4}{R_{3,eq}+R_4}\right)\left(1+\frac{R_2}{R_{1,eq}}\right),$$

where $R_{1,eq} = R_1 + R_{\text{left}}$ and $R_{3,eq} = R_3 + R_{\text{right}}$.

Therefore, we can just *fold in* the effect of R_{left} and R_{right} into these. For instance, we want to set $R_{3,eq} = 1\,\text{k}\Omega$. Now, we can actually make $R_3 = R_{3,eq} - 3\,\Omega = 997\,\Omega$ and $R_1 = R_{1,eq} - 3\,\Omega = 997\,\Omega$.

Give yourself full credit even if you didn't notice this, but keep this in mind!

Bonus: Can you now see why we wanted to keep R_1 and R_3 in the order of kΩ or larger?

5. PRACTICE: Challenge Problem: Average

The circuit in Figure 1 below operates in time steps k, as illustrated in Figure 2. Each step is of duration T. During each step, switch S_2 is opened, and then switch S_1 is closed immediately afterwards. Then, after time $T/2$, switch S_1 is opened just before switch S_2 is closed.

At the end of each time step the input $V_{in}(kT)$ changes and the process repeats.

Derive an expression for $V_2(kT)$ as a function of $V_{in}(kT)$. Use $C_1 = pC_0$ and $C_2 = (1-p)C_0$.

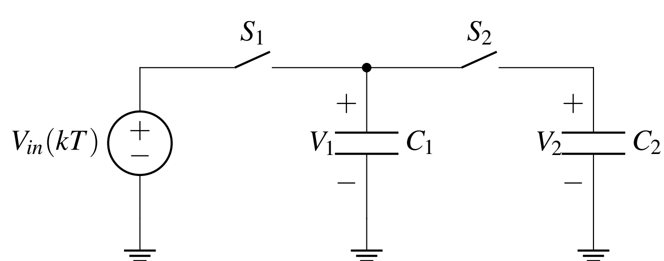


Figure 1: Averaging circuit

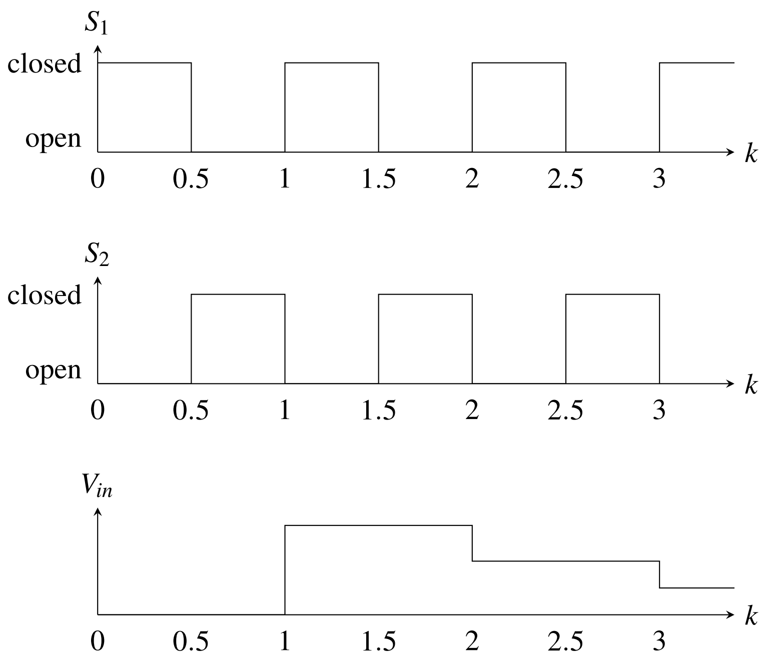


Figure 2: Timing diagram

Solution: Consider the first half of each time step. When switch S_2 is opened, no current can flow across C_2, so its voltage V_2 will remain constant from the previous time step. Then, when switch S_1 is closed, C_1 will be placed in parallel with $V_{in}(kT)$, so it will have a voltage of $V_{in}(kT)$ across its two plates. Thus, the charge on the top plate of C_1 will be $Q = C_1 V_{in}(kT)$ for the first half of the every time step.

Then, when switch S_1 is opened and then S_2 closed, C_1 and C_2 will be placed in parallel with each other, so the charge can flow between the top plate of C_1 and the top plate of C_2.

Notice that the circuit is simply sampling the input during the first half of each cycle and then averaging during the second half of each cycle. Let's see what happens if we apply charge conservation between the first and second half of the first cycle of operation:

$$\begin{aligned}
Q_1(1) + Q_2(1) &= Q_1(0.5) \\
(C_1 + C_2)V_2(1) &= C_1 V_{in}(0.5) \\
V_2(1) &= \frac{C_1}{C_1 + C_2} V_{in}(0.5) \\
V_2(1) &= \frac{pC_0 V_{in}(0.5)}{C_0} = pV_{in}(0.5)
\end{aligned}$$

When we open switch S_2 the charge on C_2 will be preserved, hence the voltage on V_2 will remain unchanged for $k = 1$ to $k = 1.5$ - the circuit has memory! Applying again charge conservation between the first and second half of the second cycle we will get:

$$\begin{aligned}
Q_1(2)+Q_2(2) &= Q_1(1.5)+Q_2(1.5)\\
(C_1+C_2)V_2(2) &= C_1V_{in}(1.5)+C_2V_2(1.5) = C_1V_{in}(1.5)+C_2V_2(1)\\
V_2(2) &= \frac{C_1V_{in}(1.5)+C_2V_2(1)}{C_1+C_2}\\
V_2(2) &= \frac{pC_0V_{in}(1.5)+(1-p)C_0V_2(1)}{C_0}\\
V_2(2) &= pV_{in}(1.5)+(1-p)pV_{in}(0.5)
\end{aligned}$$

A pattern has already started forming - we need to perform a final step of induction to get:

$$V_2(kT) = pV_{in}((k-0.5)T)+(1-p)pV_{in}((k-1.5)T)+(1-p)p^2V_{in}((k-2.5)T)+...+(1-p)^{k-1}V_{in}(0.5T)$$

The circuit calculates an “average” with bigger weights for the most recent values of V_{in} and less weight on the older ones. Pretty cool right?

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6. MANDATORY - Not in scope for MT2: Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality states that for two vectors $\vec{v},\vec{w}\in\mathbb{R}^n$:

$$\langle\vec{v},\vec{w}\rangle = \vec{v}^T\vec{w} \leq \|\vec{v}\|\cdot\|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^2$.

Take two vectors: $\vec{v} = r\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $\vec{w} = t\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}$, where $r>0$, $t>0$, θ, and ϕ are scalars. Make sure you understand why any vector in $\mathbb{R}^2$ can be expressed this way and why it is acceptable to restrict $r,t>0$.

(a) In terms of some or all of the variables r, t, θ, and ϕ, what are $\|\vec{v}\|$ and $\|\vec{w}\|$?

Solution: We use the trig identity $\cos^2 x+\sin^2 x = 1$ to show:

$$\begin{aligned}
\|\vec{v}\| &= \sqrt{v_1^2+v_2^2}\\
&= \sqrt{r^2\cos^2\theta+r^2\sin^2\theta}\\
&= r
\end{aligned}$$

Similarly, $\|\vec{w}\| = t$.

(b) In terms of some or all of the variables r, t, θ, and ϕ, what is $\langle\vec{v},\vec{w}\rangle$?

Solution: We use the trig identity $\cos(x)\cos(y)+\sin(x)\sin(y) = \cos(x-y)$ to show:

$$\begin{aligned}
\langle\vec{v},\vec{w}\rangle &= (r\cos\theta)(t\cos\phi)+(r\sin\theta)(t\sin\phi)\\
&= r\cdot t(\cos\theta\cos\phi+\sin\theta\sin\phi)\\
&= r\cdot t\cos(\theta-\phi)
\end{aligned}$$

(c) Show that the Cauchy-Schwarz inequality holds for any two vectors in $\mathbb{R}^2$.

Solution: We use the fact that $\cos x \leq 1$ to show:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= r \cdot t \cos(\theta - \phi) \\ &= \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) \\ &\leq \|\vec{v}\| \|\vec{w}\|\end{aligned}$$

(d) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \cdot \|\vec{w}\|$?

Solution:

$$\begin{aligned}\langle \vec{v}, \vec{w} \rangle &= \|\vec{v}\| \|\vec{w}\| \\ \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi) &= \|\vec{v}\| \|\vec{w}\| \\ \cos(\theta - \phi) &= 1 \\ \theta - \phi &= 0\end{aligned}$$

We see that the equality holds when the angle between the two vectors is zero or, equivalently, when the vectors are linearly dependent.

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

