EECS 16A Designing Information Devices and Systems I
Summer 2020 Homework 4B

This homework is due Sunday, July 26, 2020, at 23:59.
Self-grades are due Wednesday, July 29, 2020, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

Homework Learning Goals: *The objective of this homework is to familiarize you with capacitive modeling and measurement, as well as the use of capacitors as batteries. It also introduces circuit design using Op-Amps in negative feedback.*

1. Capacitive Touchscreen

The model for a capacitive touchscreen can be seen in Figure 1. See Table 1 for values of the dimensions. The green area represents the contact area of the finger with the top insulator. It has dimensions w_2 x d_1.

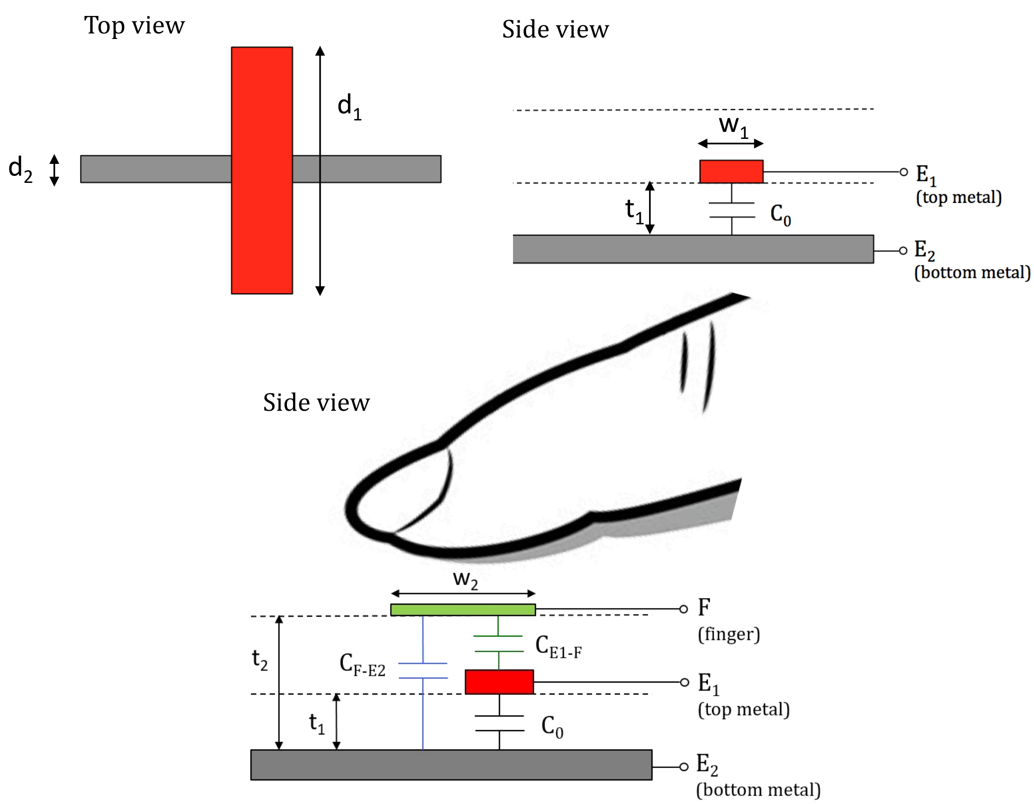


Figure 1: Model of capacitive touchscreen.

Table 1: Touchscreen Dimension Values

d_1	10 mm
d_2	1 mm
t_1	2 mm
t_2	4 mm
w_1	1 mm
w_2	2 mm

(a) Draw the equivalent circuit of the touchscreen that contains the nodes F, E_1, and E_2 when there no finger present and when there is a finger present. Express the capacitance values in terms of C_0, C_{F-E1}, and C_{F-E2}.

Solution:

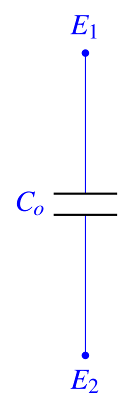


Figure 2: Touchscreen circuit with no finger present.

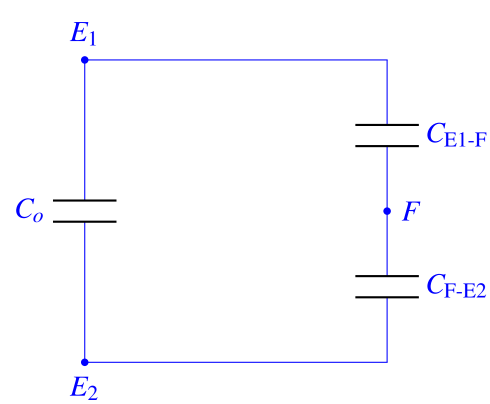


Figure 3: Touchscreen circuit with a finger present.

(b) What are the values of C_0, C_{F-E1}, and C_{F-E2}? Assume that the insulating material has a permittivity of $\varepsilon = 4.43 * 10^{-11} F/m$ and that the thickness of the metal layers is small compared to t_1.

Solution:

$$C_0 = \varepsilon \frac{d_2 w_1}{t_1} = 2.215 * 10^{-14} F$$

$$C_{F-E1} = \varepsilon \frac{d_1 w_1}{t_2 - t_1} = 2.215 * 10^{-13} F$$

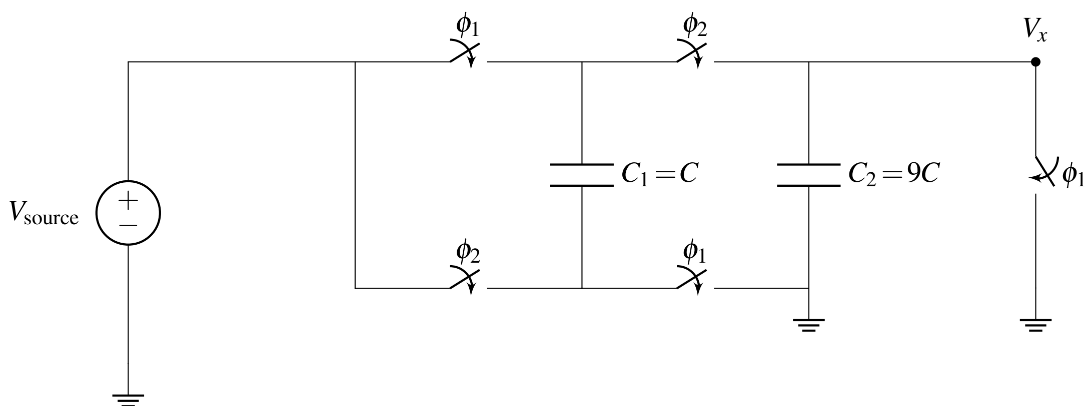
$$C_{F-E2} = \varepsilon \frac{d_2(w_2 - w_1)}{t_2} = 1.108 * 10^{-14} F$$

(c) What is the difference in effective capacitance between the two metal plates (nodes E_1 and E_2) when a finger is present?

Solution: The effective capacitance between the two plates is $C_0 = 2.215 * 10^{-14} F$ when there is no finger. When there is a finger, we have C_0 in parallel with a series combination of C_{F-E1} and C_{F-E2}, giving an additional capacitance $C_{F-E1}||C_{F-E2} = 1.055 * 10^{-14} F$ when a finger is present. Therefore, the total effective capacitance is: $3.270 * 10^{-14} F$

2. Charge Sharing

Consider the following circuit:



In the first phase, all of the switches labeled ϕ_1 will be closed and all switches labeled ϕ_2 will be open. In the second phase, all switches labeled ϕ_1 are opened and all switches labeled ϕ_2 are closed.

(a) Draw the polarity of the voltage (using + and − signs) across the two capacitors C_1 and C_2. (It doesn't matter which terminal you label + or −; just remember to keep these consistent through phase 1 and 2!)

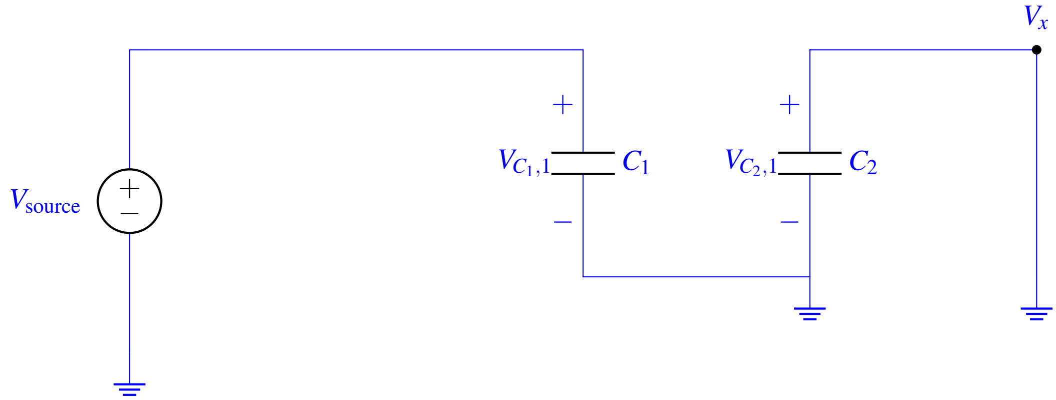
Solution:

One way of marking the polarities is + on the top plate and − on the bottom plate of both C_1 and C_2. Let's call the voltage drop across C_1 V_{C_1} and across C_2 V_{C_2}.

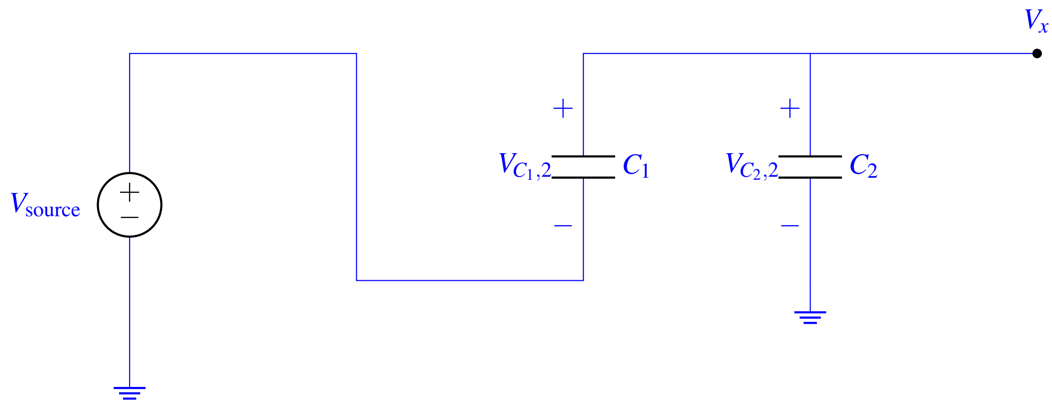
(b) Draw the circuit in the first phase and in the second phase. Keep your polarity from part (a) in mind.

Solution:

Phase 1



Phase 2



In phase 1, all the switches marked as ϕ_1 are closed and switches marked as ϕ_2 are open. In phase 2, all the switches marked as ϕ_2 are closed and switches marked as ϕ_1 are open. Draw both the circuits separately, side by side, with the switches in their respective positions.

(c) Find the voltage across and the charge on C_1 and C_2 in phase 1. Be sure to keep the polarities of the voltages the same!

Solution:

In phase 1,

$$V_{C_1,1} = V_{\text{source}} - 0 = V_{\text{source}}$$

and

$$V_{C_2,1} = 0 - 0 = 0$$

Solution:

Next, we find the charge on each capacitor:

$$Q_{C_1,1} = V_{C_1,1}C_1 = CV_{\text{source}}$$

Note that the positive plate has a charge of $+CV_{\text{source}}$, while the negative plate has a charge of $-CV_{\text{source}}$.

$$Q_{C_2,1} = V_{C_2,1}C_2 = 0$$

(d) Now, in the second phase, find the voltage V_x.

Solution:

Where is charge conserved? To answer this, look at the top plates of C_1 and C_2. In phase 2, they are both "floating" because they are not connected to V_{source} or ground. And in phase 1, they are not connected to each other, but in phase 2, they are connected by the switch. Therefore, in phase 2, the charges on the top plates of C_1 and C_2 will be *shared*, or distributed, because they simply cannot go anywhere else. The total charge will remain the same as in phase 1. Let's find the voltages across C_1 and C_2 in phase 2 (same polarities as in phase 1!):

$$V_{C_1,2} = V_x - V_{\text{source}}$$

and

$$V_{C_2,2} = V_x$$

Now, let's find the charge stored in top plates of C_1 and C_2:

$$Q_{C_1,2} = C(V_x - V_{\text{source}})$$

and

$$Q_{C_2,2} = 9CV_x$$

Next, let's write the equation for charge conservation:

$$Q_{C_1,1} + Q_{C_2,1} = Q_{C_1,2} + Q_{C_2,2},$$

giving

$$CV_{\text{source}} + 0 = C(V_x - V_{\text{source}}) + 9CV_x,$$

which results in

$$V_x = \frac{V_{\text{source}}}{5}.$$

(e) If the capacitor C_2 did not exist (i.e. had a capacitance of $0\,\text{F}$), what would the voltage V_x be?

Solution:

We could always go back to the equations above, plug in $C_2 = 0$, and derive $V_x = 2V_{\text{source}}$. It might be worthwhile to go over what this means for the circuit, though. If $C_2 = 0\,\text{F}$, the capacitor is actually an open circuit. (*Why?*) So we can pretend, as the question says, that C_2 does not exist. In phase 1, as before, C_1 has a voltage drop of V_{source} across it (from top to bottom) and is charged up to CV_{source}. Now, in phase 2, the top plate of C_1 is left dangling (floating). This means that the charge on the top plate of C_1 is going to be the same just like the charge on the bottom plate. We will therefore get

$$V_x = V_{\text{source}} - (-V_{\text{source}}) = 2V_{\text{source}}$$

Recipe for charge sharing:

i. Label the voltages across all the capacitors. Choose whichever direction (polarity) you want foreach capacitor - this means you can mark any one of the plates with the "+" sign, and then you can mark the other plate with the "-" sign. Just make sure you stay consistent with this polarity across phases.

ii. Draw the equivalent circuit during both phases (Phase 1: ϕ_1 closed, ϕ_2 open - Phase 2: ϕ_1 open, ϕ_2 closed). Also, label all node voltages on the circuit for both phases. No need to try and maintain the same names, since certain nodes of the Phase 1 circuit might be merged or split in Phase 2.

iii. Identify all "floating" nodes in your circuit during Phase 2. A floating node is a node out of or into which no charge can flow. You can identify those nodes as the nodes connected only to capacitor plates, op-amp inputs or comparator inputs. These will be the nodes where we apply charge sharing.

iv. Pick a floating node from the ones you found in Step 3 and identify all capacitor plates connected to that node during Phase 2. Then, calculate the charge on each of these plates during Phase 1. To do so, identify all nodes in your circuit during Phase 1. Label all node voltages, and write the voltages across each capacitor as functions of node voltages (Step 2 should help you with that). Do this according to the polarities you have selected. Then the charge is found as $Q = CV_C$ (where V_C is the voltage across a capacitor).

v. Find the total charge on each of the floating nodes during Phase 2 as a function of node voltages.Use the same process as in Step 4, but this time using the node voltages during Phase 2 to write the voltages across each capacitor. Make sure you kept the polarity same and pay attention to the sign of each plate.

vi. Equate the total charge calculated in Phase 1 (Step 4) to the total charge calculated in Phase 2 (Step 5) (charge conservation).

vii. Repeat Steps 4-6 for every floating node. This will give you one equation per floating node (i.e. if you have m floating nodes you will have m equations). You can then solve the system of equations to find the node voltages during Phase 2 (unknowns). It should have a unique solution!

As you probably noticed this exercise was architected in a way that basically walked you through this recipe used to approach charge sharing problems.

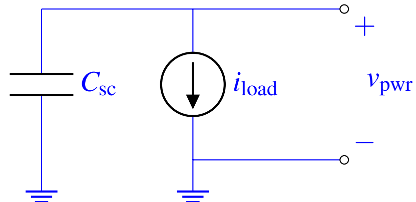
3. Super-Capacitors

In order to enable small devices for the "Internet of Things" (IoT), many companies and researchers are currently exploring alternative means of storing and delivering electrical power to the electronics within these devices. One example of these are "super-capacitors" - the devices generally behave just like a "normal" capacitor but have been engineered to have extremely high values of capacitance relative to other devices that fit in to the same physical volume. They can function as a power supply for low power applications such as IoT devices and have the advantage that they can be charged and discharged many times without losing maximum charge capacity. That property makes super-capacitors suitable to store energy from intermittent power sources such as those from energy harvesting. Suppose you are tasked with designing a power supply with a super-capacitor in an IoT device.

(a) Assuming that your electronic device (load) can be modeled as a constant current source with a value of i_{load}, draw circuit models for your device using super-capacitors as the power supply with the following configurations:

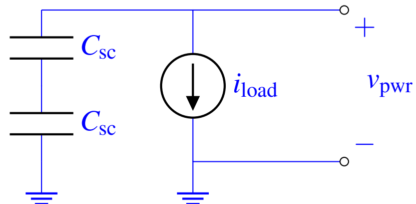
- Config 1: a single super-capacitor as the power supply

  Solution:

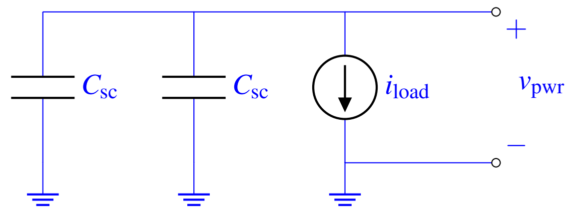


- Config 2: two super-capacitors stacked in series as the power supply

  Solution:



- Config 3: two super-capacitors connected in parallel as the power supply
  Solution:



(b) If each super-capacitor is charged to an initial voltage v_{init} and has a capacitance of C_{sc}, for each of the three configurations above, write an expression for the voltage supplied to your electronic device as a function of time after the device has been activated (i.e. connected to the super-capacitor(s)).

Solution:

Let the initial voltage each super capacitor is charged to be v_{init}. We'll now consider the three situations:

- Config 1: **Single super capacitor**
  In this case, the initial voltage that the super capacitor provides is v_{init}, and the initial charge stored in it is then given by $Q_{init} = v_{init}C_{sc}$. Let the voltage at any time, t be defined by $v_{pwr}(t)$. The charge drained by the constant current source, i_{load} in time t is given by $i_{load}t$. The effective charge stored in the capacitor after time t is given by $Q(t) = Q_{init} - i_{load}t$. Therefore,

$$\begin{aligned} v_{pwr}(t) &= \frac{Q(t)}{C_{sc}} \\ &= \frac{Q_{init} - i_{load}t}{C_{sc}} \\ &= \frac{v_{init}C_{sc} - i_{load}t}{C_{sc}} \\ &= v_{init} - \frac{i_{load}t}{C_{sc}} \end{aligned}$$

- Config 2: **Two super capacitors in series**
  In this case, the initial voltage that the effective super capacitor provides is $2v_{init}$, and the effective capacitance is $C_{eq} = \frac{C_{sc}}{2}$. Then, the initial effective charge stored in them is then given by $Q_{init} = 2v_{init}C_{eq} = 2v_{init}\frac{C_{sc}}{2} = v_{init}C_{sc}$. Let the voltage at any time, t be defined by $v_{pwr}(t)$. The charge drained by the constant current source in time t is given by $i_{load}t$. The effective charge stored in the combination after time t is given by $Q(t) = Q_{init} - i_{load}t$. Therefore,

$$\begin{aligned} v_{pwr}(t) &= \frac{Q(t)}{C_{eq}} \\ &= \frac{Q_{init} - i_{load}t}{C_{eq}} \\ &= \frac{v_{init}C_{sc} - i_{load}t}{C_{eq}} \\ &= 2v_{init} - \frac{2i_{load}t}{C_{sc}} \end{aligned}$$

- Config 3: **Two super capacitors in parallel**
  In this case, the initial voltage that the effective super capacitor provides is v_{init}, and the effective capacitance is $C_{eq} = 2C_{sc}$. Then, the initial effective charge stored in them is given by $Q_{init} = v_{init}C_{eq} = 2v_{init}C_{sc}$. Let the voltage at any time t be defined by $v_{pwr}(t)$. The charge drained by the

constant current source in time t is given by $i_{\text{load}}t$. The effective charge stored in the combination after time t is given by $Q(t) = Q_{\text{init}} - i_{\text{load}}t$. Therefore,

$$\begin{aligned} v_{\text{pwr}}(t) &= \frac{Q(t)}{C_{\text{eq}}} \\ &= \frac{Q_{\text{init}} - i_{\text{load}}t}{C_{\text{eq}}} \\ &= \frac{2v_{\text{init}}C_{\text{sc}} - i_{\text{load}}t}{C_{\text{eq}}} \\ &= v_{\text{init}} - \frac{i_{\text{load}}t}{2C_{\text{sc}}} \end{aligned}$$

(c) Now let's assume that your electronic device requires some minimum voltage v_{min} in order to function properly. For each of the three super-capacitor configurations, write an expression for the lifetime of the device.

Solution:

The lifetime of a device is the time it takes for the $v_{\text{pwr}}(t)$ to hit the threshold v_{min}. For each of the three configurations, let's find out their lifetime (denoted by t_0):

- Config 1: **Single super capacitor**
  Let us calculate at what time $t = t_0$, $v_{\text{pwr}}(t)$ equals v_{min}. We know from the previous part that

  $$v_{\text{pwr}}(t) = v_{\text{init}} - \frac{i_{\text{load}}t}{C_{\text{sc}}}.$$

  Substituting $v_{\text{pwr}}(t) = v_{\text{min}}$, we get

  $$t_0 = \frac{(v_{\text{init}} - v_{\text{min}})C_{\text{sc}}}{i_{\text{load}}}.$$

- Config 2: **Two super capacitors in series**
  Let us calculate at what time $t = t_0$, $v_{\text{pwr}}(t)$ equals v_{min}. We know from the previous part that

  $$v_{\text{pwr}}(t) = 2v_{\text{init}} - \frac{2i_{\text{load}}t}{C_{\text{sc}}}.$$

  Substituting $v_{\text{pwr}}(t) = v_{\text{min}}$, we get

  $$t_0 = \frac{(2v_{\text{init}} - v_{\text{min}})C_{\text{sc}}}{2i_{\text{load}}}.$$

- Config 3: **Two super capacitors in parallel**
  Let us calculate at what time $t = t_0$, $v_{\text{pwr}}(t)$ equals v_{min}. We know from the previous part that

  $$v_{\text{pwr}}(t) = v_{\text{init}} - \frac{i_{\text{load}}t}{2C_{\text{sc}}}.$$

  Substituting $v_{\text{pwr}}(t) = v_{\text{min}}$, we get

  $$t_0 = \frac{(v_{\text{init}} - v_{\text{min}})2C_{\text{sc}}}{i_{\text{load}}}.$$

Note: We could have also figured it out by finding out how much charge needs to be removed to cause the voltage at the effective capacitance to drop to v_{min}. Thus, we have

$$\Delta Q = (v_{\text{init}} - v_{\text{min}})C_{\text{eq}},$$

which gives us

$$t_0 = \frac{\Delta Q}{i_{\text{load}}}.$$

(d) Assume that a single super-capacitor doesn't provide you sufficient lifetime and so you have to spend the extra money (and device volume) for another super-capacitor. You consider the two following configurations:

- Config 2: two super-capacitors stacked in series
- Config 3: two super-capacitors connected in parallel

When is Config 3 (parallel) better than Config 2 (series)? Your answer should involve conditions on v_{init} and v_{min}.

Solution:

It is not obvious from the previous part whether configuration 2 or 3 will provide a longer lifetime. In fact, it depends on what v_{init} is with respect to v_{min}. Let us now see what conditions we need on v_{init}, such that the parallel configuration provides a longer lifetime, i.e., $t_{0,\text{ parallel}} > t_{0,\text{ series}}$. From the previous part, we get

$$\frac{(v_{\text{init}} - v_{\text{min}})2C_{\text{sc}}}{i_{\text{load}}} > \frac{(2v_{\text{init}} - v_{\text{min}})C_{\text{sc}}}{2i_{\text{load}}}$$
$$2(v_{\text{init}} - v_{\text{min}}) > v_{\text{init}} - \frac{v_{\text{min}}}{2}$$
$$v_{\text{init}} > \frac{3}{2}v_{\text{min}}$$

Thus, we see that when $v_{\text{init}} > \frac{3}{2}v_{\text{min}}$, the parallel configuration is better; otherwise the series configuration is better.

(e) Calculate the amount of energy delivered by the super-capacitors in Config 3 (parallel) over the device's lifetime.

Solution: The initial energy stored by the super-capacitors is $\frac{1}{2}C_{eq}V^2 = C_{sc}v_{init}^2$. At the end of the lifetime, the super-capacitor voltage is v_{min}, so the energy stored in the super-capacitors is $C_{sc}v_{min}^2$. The amount of energy delivered is the difference of the initial and final energies, so it is $C_{sc}(v_{init}^2 - v_{min}^2) = C_{sc}(v_{init} + v_{min})(v_{init} - v_{min})$.

4. Op-Amp in Negative Feedback

In this question, we analyze op amp circuits that have finite gain. We replace the op amp with its circuit model with parameterized gain and observe the gain's effect on terminal and output voltages as the gain approaches infinity. Figure 4 shows the equivalent model of the op-amp.

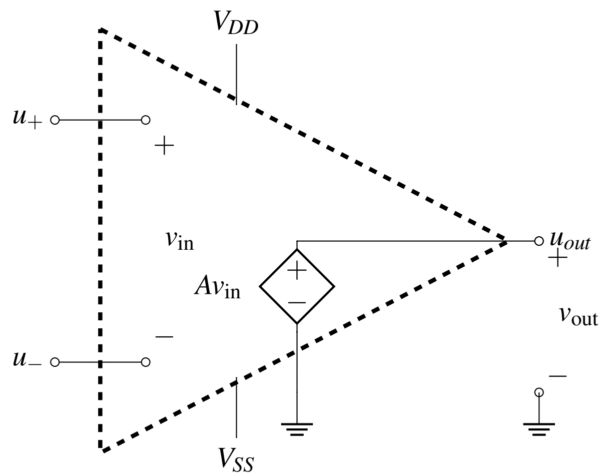


Figure 4: Op-amp model

(a) Consider the circuit shown in Figure 5. Assume that the op amp is ideal ($A \to \infty$) for parts (a) through (e). What is $u_+ - u_-$?

Solution: For ideal op amp circuits in negative feedback, the voltage at the two terminals must be equal, so $u_+ - u_- = 0$.

(b) Find v_x as a function of v_{out}.

Solution: We see that v_x is the middle node of a voltage divider, so $v_x = v_{out}\frac{R_1}{R_1+R_2}$.

(c) What is the current flowing through R_2 as a function of v_s?

Solution: We know from part (a) that $v_x = v_s$. The current flowing through R_1 is $I_{R_1} = \frac{v_s}{R_1}$. This current also flows through R_2.

(d) Find v_{out} as a function of v_s.

Solution: Using the answer from the previous part, $v_{out} = v_s + R_2 I_{R_1} = v_s + R_2\frac{v_s}{R_1} = v_s\left(\frac{R_1+R_2}{R_1}\right)$.

(e) What is the current i_L through the load resistor R? Give your answer in terms of v_{out}.

Solution: The current i_L through the load is $\frac{v_{out}}{R}$.

(f) We will now examine what happens when A is not ∞. Draw an equivalent circuit by replacing the op-amp with the op-amp model shown in Figure 4 and calculate v_{out} and v_x in terms of A, v_s, R_1, R_2 and R. Is the magnitude of v_x larger or smaller than the magnitude of v_s? Do these values depend on R?

Solution:

This is the equivalent circuit of the op-amp:

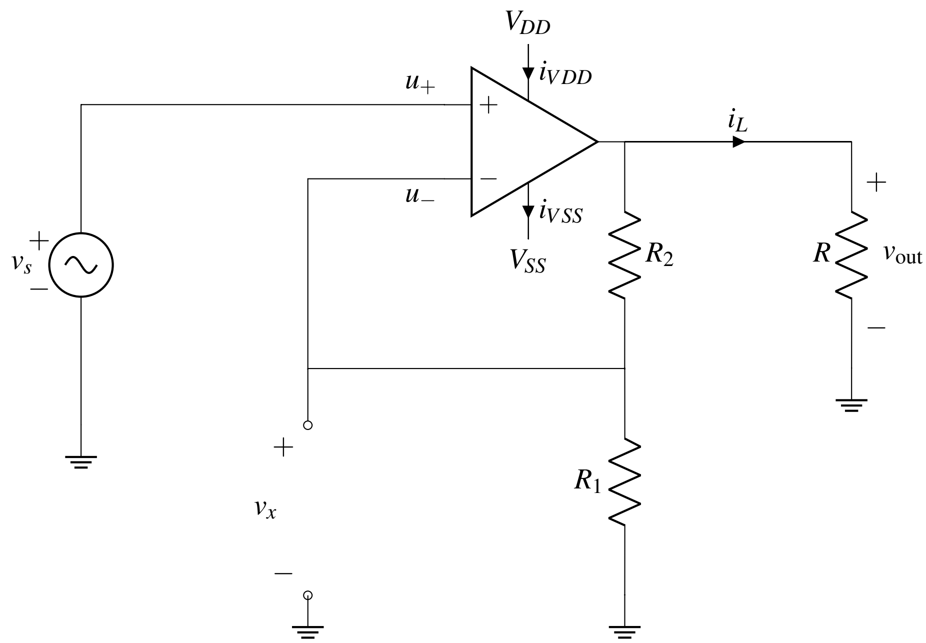
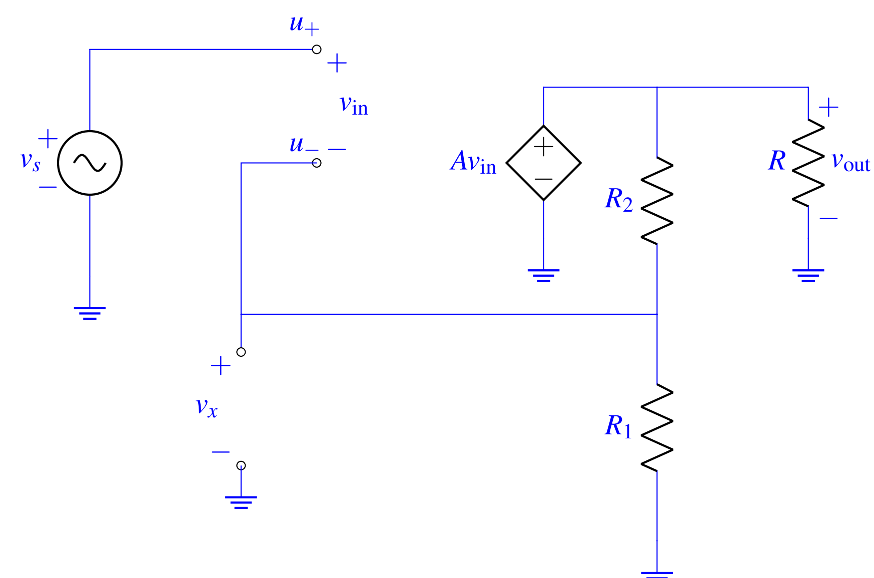


Figure 5: Non-inverting amplifier circuit



Since v_{out} is connected to the output of the op-amp, which is a voltage source, we can determine v_{out}:

$$\begin{aligned} v_{out} &= A(u_+ - u_-) \\ &= A(v_s - v_x) \end{aligned}$$

Since there is no current flowing into the op amp input terminals from nodes u_+ and u_-, R_1 and R_2

form a voltage divider and $v_x = v_{\text{out}}\left(\frac{R_1}{R_1+R_2}\right)$. Thus, substituting and solving for v_{out}:

$$v_{\text{out}} = A\left(v_s - v_{\text{out}}\frac{R_1}{R_1+R_2}\right)$$
$$v_{\text{out}} = v_s\left(\frac{1}{\frac{R_1}{R_1+R_2}+\frac{1}{A}}\right)$$

Knowing v_{out}, we can find v_x:

$$v_x = \frac{v_s}{1+\frac{R_1+R_2}{AR_1}}$$

Notice that v_x is slightly smaller than v_s, meaning that in equilibrium in the non-ideal case, v_+ and v_- are not equal. v_{out} and v_x do not depend on R, which means that we can treat v_{out} as a voltage source that supplies a constant voltage independent of the load R.

(g) Using your solution to the previous part, calculate the limits of v_{out} and v_x as $A \to \infty$. Do you get the same answer as in part (d)?

Solution:

As $A \to \infty$, the fraction $\frac{1}{A} \to 0$, so

$$v_{\text{out}} = v_s\left(\frac{1}{\frac{R_1}{R_1+R_2}+\frac{1}{A}}\right)$$

converges to

$$v_s\left(\frac{1}{\frac{R_1}{R_1+R_2}+0}\right) = v_s\left(\frac{R_1+R_2}{R_1}\right).$$

Therefore, the limits as $A \to \infty$ are:

$$v_{\text{out}} \to v_s\left(\frac{R_1+R_2}{R_1}\right)$$
$$v_x \to v_s$$

If we observe the op amp is in negative feedback, we can apply the fact that $u_+ = u_-$. We get $v_x = v_s$. Then the current i flowing through R_1 to ground is $\frac{v_s}{R_1}$. By KCL, this same current flows through R_2 since no current flows into the negative input terminal of the op amp (u_-). Thus, the voltage drop across R_2 is $v_{\text{out}} - v_x = i \cdot R_2 = v_s\left(\frac{R_2}{R_1}\right)$. Therefore, $v_{\text{out}} = v_s + v_s\left(\frac{R_2}{R_1}\right) = v_s\left(\frac{R_1+R_2}{R_1}\right)$. The answers are the same if you take the limit as $A \to \infty$.

(h) Now you want to make a circuit whose gain is nominally $G_{nom} = \frac{v_{out}}{v_s} = 4$ with a minimum error of 1% (a minimum gain of $G_{min} = 3.96$). What is the minimum required gain of the amplifier A_{min} to achieve that specification?

Solution: From the previous part, $v_{\text{out}} = v_s\left(\frac{1}{\frac{R_1}{R_1+R_2}+\frac{1}{A}}\right)$. After algebraic manipulations, we get

$$v_{out} = v_s\left(\frac{A(R_1+R_2)}{R_1+R_2+AR_1}\right)$$

We are interested in the op amp's minimum gain A_{min}, which gives us the circuit's corresponding minimum gain G_{min}.

We define the minimum (actual - i.e. corresponding to a non-infinite $A = A_{min}$) gain as: $G_{min} = \frac{v_{out}}{v_s}$
We also define the nominal (ideal - i.e. corresponding to an infinite A) gain as: $G_{nom} = 1 + \frac{R_2}{R_1}$.
Rewriting A_{min} in terms of G_{nom} and G_{min} gives:

$$\begin{aligned} A_{min} &= \frac{G_{min}G_{nom}}{G_{nom} - G_{min}} \\ &= 396 \end{aligned}$$

Notice that the op amp's minimum gain is independent of the resistor values. In general, if we wanted an error of less than ε, then the following will approximately hold: $\frac{A_{min}}{G_{nom}} > \frac{1}{\varepsilon}$.

5. **Cool For The Summer**

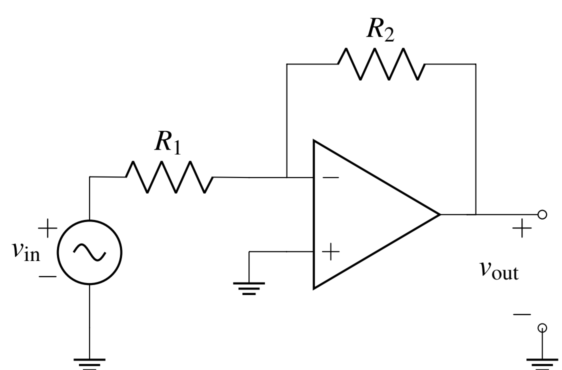
You and a friend want to make a box that helps control an air conditioning unit using both your inputs. You both have individual dials where you can set a control voltage: input of 0 means that you want to leave the temperature as it is. Negative voltage input would mean that you want to reduce the temperature. (It's hot, so we will assume that you never want to increase the temperature – so, we're not talking about a Berkeley summer...)

Your air conditioning unit, however, responds to positive voltages. The higher the magnitude of the voltage, the stronger it runs. At zero, it is off. You also need a system that sums up both you and your friend's control inputs.

Therefore, you need a box that is **an inverting summer** – *it outputs a weighted sum of two voltages where the weights are both negative*. The sum is weighted because each of you has your own subjective sense of how much to turn the dial down, so you need to compensate for this.

This problem walks you through designing this inverting summer using an op-amp.

(a) As a first step, derive v_{out} in terms of R_2, R_1, v_{in}.



Solution: First, we need to check that the amplifier is in negative feedback. In other words, if the negative input terminal is moved upward, the feedback needs to move it back downward. Going around the loop:

- We move the negative input of the op amp upward
- The output of the amplifier moves downward
- The negative input moves downward with it

The important thing here is that the result of the initial stimulus needs to go in the opposite direction of the initial stimulus! Thus, we've confirmed that the amplifier is in negative feedback.

Second, we perform KCL.

$$\frac{v_{\text{in}} - u_-}{R_1} + \frac{v_{\text{out}} - u_-}{R_2} = 0$$

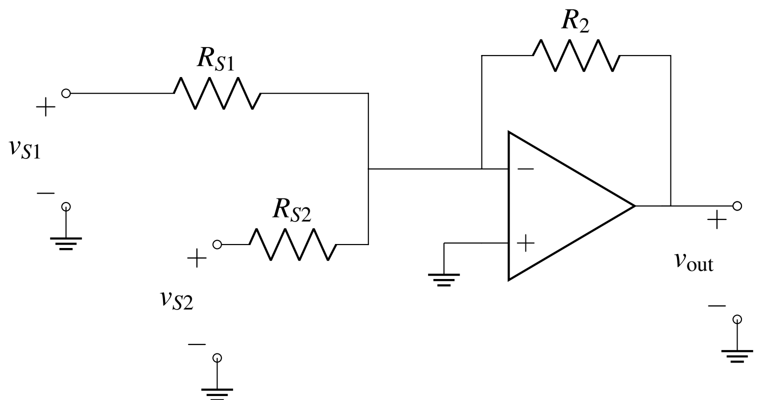
Since we're in negative feedback, we can apply the golden rules. From those, we know the voltages at the negative and positive input terminals of the amplifier—u_- and u_+, respectively—are held at the same voltage. In other words, $u_+ = u_- = 0\text{V}$.

$$\frac{v_{\text{in}}}{R_1} + \frac{v_{\text{out}}}{R_2} = 0$$

$$v_{\text{out}} = v_{\text{in}}\left(-\frac{R_2}{R_1}\right)$$

The general inverting amplifier shown above has a voltage gain $v_{\text{out}} = -\frac{R_2}{R_1}v_{\text{in}}$.

(b) Now we will add a second input to this circuit as shown below. Find v_{out} in terms of v_{S1}, v_{S2}, R_{S1}, R_{S2} and R_2.



Solution:

Method 1: Superposition

We can find the overall voltage gain of this amplifier using superposition. When v_{S1} is on, we can ignore R_{S2}. From the Golden Rules, we know that the voltage at the $-$ terminal of the op-amp must be equal to the voltage at the $+$ terminal. Thus, the voltage across R_{S2} is 0 V. Now apply the equation from part (a) $v_{\text{out}} = -\frac{R_2}{R_{S1}}v_{S1}$. Similarly, when v_{S2} is on, we get $v_{\text{out}} = -\frac{R_2}{R_{S2}}v_{S2}$. Combining the two equations, we get $v_{\text{out}} = -R_2\left(\frac{v_{S1}}{R_{S1}} + \frac{v_{S2}}{R_{S2}}\right)$.

Method 2: KCL without superposition

The following analysis is also correct and arrives at the same conclusion. According to the golden rules, $u_- = u_+ = 0\text{V}$, so we can write a single equation and solve:

$$\frac{v_{S1}}{R_{S1}} + \frac{v_{S2}}{R_{S2}} + \frac{v_{\text{out}}}{R_2} = 0$$

$$v_{\text{out}} = -v_{S1}\left(\frac{R_2}{R_{S1}}\right) - v_{S2}\left(\frac{R_2}{R_{S2}}\right)$$

(c) Let's suppose that you want $v_{\text{out}} = -\left(\frac{1}{4}v_{S1} + 2v_{S2}\right)$ where v_{S1} and v_{S2} represent the input voltages from you and your friend. Select resistor values such that the circuit implements this desired relationship.

Solution: Using the configuration from the previous part, the conditions which need to be satisfied are:

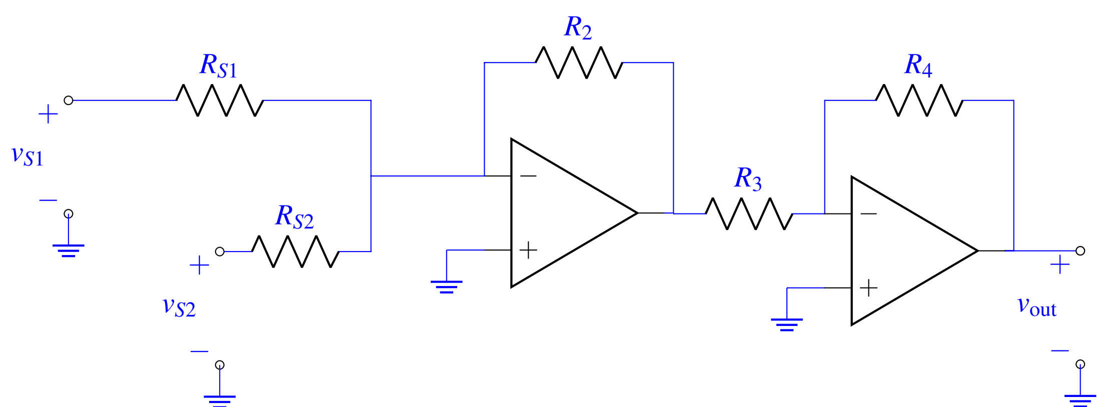
- $\frac{R_2}{R_{S1}} = \frac{1}{4}$
- $\frac{R_2}{R_{S2}} = 2$

One possible set of values is $R_2 = 2\,\text{k}\Omega$, $R_{S1} = 8\,\text{k}\Omega$, and $R_{S2} = 1\,\text{k}\Omega$, but any combination of resistors which satisfies the conditions listed above are valid solutions.

(d) Now suppose that you have a new AC unit that you want to use with your control inputs v_{S1} and v_{S2}. This unit, however, functions opposite to the previous unit; it responds to negative voltages. The higher the magnitude of the negative voltage, the stronger the AC runs.

Now design a circuit that *outputs a weighted sum of two control input voltages where both weights are positive*. Specifically, add another op-amp based circuit to your circuit in part (b), so that you invert the output of the circuit from part (b).

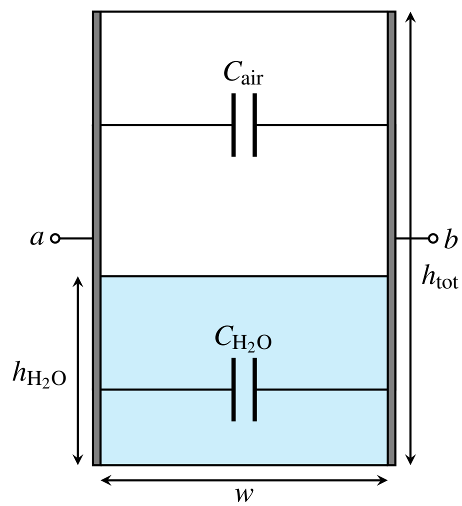
Solution:



Here, we add another inverting op-amp stage with a voltage gain of 1, and we can pick any equal-valued resistors for R_3 and R_4.

6. It's finally raining!

A lettuce farmer in Salinas Valley has grown tired of weather.com's imprecise rain measurements. Therefore, they decided to take matters into their own hands by building a rain sensor. They placed a rectangular tank outside and attached two metal plates to two opposite sides in an effort to make a capacitor whose capacitance varies with the amount of water inside.



The width and length of the tank are both w (i.e., the base is square) and the height of the tank is h_{tot}.

(a) What is the capacitance between terminals a and b when the tank is full? What about when it is empty? *Note:* the permittivity of air is ε, and the permittivity of rainwater is 81ε.

Solution:

Capacitance of parallel plates is governed by the equation:

$$C = \frac{\varepsilon A}{d},$$

where ε is the *permittivity* of the dielectric material, A is the area of the plates, and d is the distance between the plates. If we apply this to our physical structure, we find that the area of the plates are $h_{\text{tot}} \cdot w$, and the distance between the plates is w. The only difference here between a full and empty tank is the permittivity of the material between the two plates.

$$C_{\text{empty}} = \frac{\varepsilon_{\text{air}}\, h_{\text{tot}} w}{w} = \varepsilon\, h_{\text{tot}}$$

$$C_{\text{full}} = \frac{\varepsilon_{H_2O}\, h_{\text{tot}} w}{w} = 81\varepsilon\, h_{\text{tot}}$$

(b) Suppose the height of the water in the tank is h_{H_2O}. Modeling the tank as a pair of capacitors in parallel, find the total capacitance between the two plates. Call this capacitance C_{tank}.

Solution:

We can break the total capacitance into two parts. First, let's calculate the capacitance of the two plates separated by water:

$$C_{\text{water}} = \frac{\varepsilon_{H_2O}\, h_{H_2O} w}{w} = 81\varepsilon\, h_{H_2O}$$

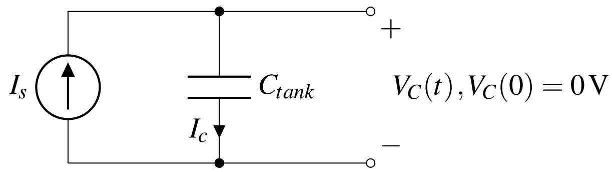
And now we can calculate the capacitance of the two plates separated by air:

$$C_{\text{air}} = \frac{\varepsilon_{\text{air}}\,(h_{\text{tot}} - h_{H_2O})\, w}{w} = \varepsilon\,(h_{\text{tot}} - h_{H_2O})$$

Because these two capacitors appear in parallel, we can simply add our two previous results to find the total equivalent capacitance:

$$C_{\text{tank}} = C_{\text{water}} + C_{\text{air}} = \varepsilon\,(h_{\text{tot}} + 80 h_{H_2O})$$

(c) After building this capacitor, the farmer consults the internet to assist them with a capacitance-measuring circuit. A fellow internet user recommends the following:



In this circuit, C_{tank} is the total tank capacitance that you calculated earlier. I_s is a known current supplied by a current source.

The suggestion is to measure V_C for a brief interval of time, and then use the difference to determine C_{tank}.

Determine $V_C(t)$, where t is the number of seconds elapsed since the start of the measurement. You should assume that before any measurements are taken, the voltage across C_{tank}, i.e. V_C, is initialized to $0\,\mathrm{V}$, i.e. $V_C(0) = 0$.

Solution: The element equation for the capacitor is:

$$I_C = C_{tank} \frac{dV_C}{dt}$$

We also know from KCL that:

$$I_C = I_s$$

Thus, we get the following differential equation for V_C:

$$\frac{dV_C}{dt} = \frac{I_s}{C_{tank}}$$

We recall that I_s and C_{tank} are constant values and the initial value of V_C is zero ($V_C(0) = 0$). Applying these facts and integrating the differential equation, we get the following equation for V_C:

$$V_C(t) = \frac{I_s}{C_{tank}} t$$

(d) Using the equation you derived for $V_C(t)$, describe how you can use this circuit to determine C_{tank} and $h_{\mathrm{H_2O}}$.

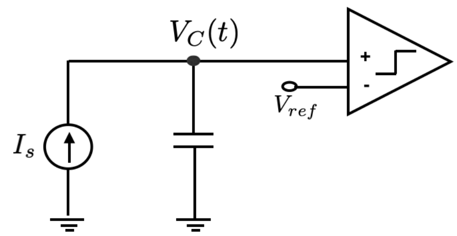
Solution: We connect the current source providing I_s A to the capacitor C_{tank}. At the same time, we can measure $V_C(t)$. After some time passes, we measure $V_C(t)$ and plug it into the following equation (assuming, as before, that $V_C(0) = 0$):

$$C_{tank} = \frac{I_s}{V_C(t)} t$$

If we know C_{tank}, we can determine $h_{\mathrm{H_2O}}$. Using the equation derived in part (b), we see that

$$h_{\mathrm{H_2O}} = \frac{C_{tank} - h_{tot}\varepsilon}{80\varepsilon}$$

(e) However, after spending some time thinking about different ways of measuring this capacitance you came up with a better idea. You decided to use the circuit proposed in part (c) along with a comparator, as show in the figure below. What you are basically interested in, is the time T_1 needed for V_c to reach V_{ref}. In order to measure time you use a timer. When voltage V_c becomes larger than V_{ref}, the comparator flips its value and you stop the timer. How would you measure in that case the value of the capacitance?



Solution: We connect the current source providing I_s A to the capacitor C_{tank}. The expression for $V_C(t)$ can be given by:

$$V_C(t) = \frac{I_s}{C_{tank}} t$$

We are interested at measuring T_1 when the V_c reached V_{ref} and the comparator flips. At T_1, V_c is equal to V_{ref}. Therefore by knowing the reference voltage V_{ref} and measuring with a timer T_1, the capacitance can be calculated by:

$$C_{tank} = \frac{I_s T_1}{V_{ref}}$$

If we know C_{tank}, we can determine $h_{\mathrm{H_2O}}$. Using the equation derived in part (b), we see that

$$h_{\mathrm{H_2O}} = \frac{C_{tank} - h_{tot}\varepsilon}{80\varepsilon}$$

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.