EECS 16A Designing Information Devices and Systems I
Summer 2020 Homework 3A

This homework is due Wednesday, July 15, 2020, at 23:59.
Self-grades are due Sunday, July 19, 2020, at 23:59.

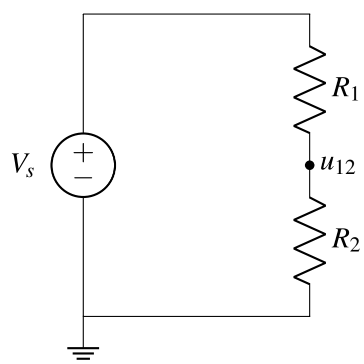
Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

Homework Learning Goals: *The objective of this homework is to introduce Node Voltage Analysis in the context of voltage divider circuits and circuits with more than one independent sources.*

1. Voltage divider

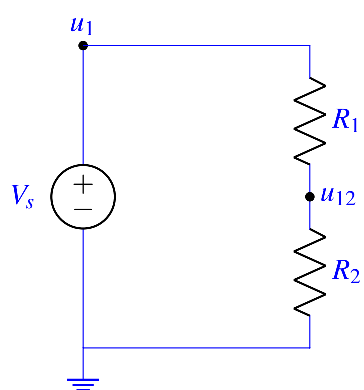
In the following parts, $V_s = 12\,\text{V}$. Choose resistance values such that the current through each element is $\leq 0.8\text{A}$.

(a) Select values for R_1 and R_2 in the circuit below such that $u_{12} = 6\,\text{V}$.

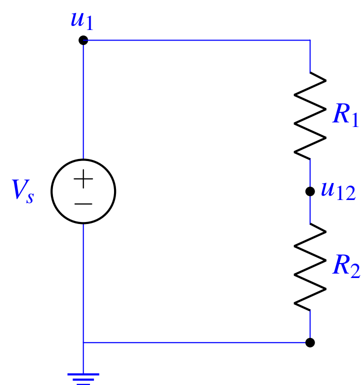


Solution: Step 1: Reference Node
We notice that the ground node has already been selected for us in the question.

Step 2: Label nodes with voltage set by sources



Step 3: Label other nodes



Step 4: Label element voltages and currents.
Let V_{R1}, V_{R2} be the voltage drop across R_1 and R_2 respectively. Let I_1 be the current between the voltage source and R_1. Let I_2 be the current from R_2 to the voltage source.
Step 5: KCL Equations
From KCL, at u_{12},

$$I_1 = I_2.$$

Step 6: Find element currents
Using Ohm's law, we have that

$$I_1 = \frac{V_{R1}}{R_1}$$

$$I_2 = \frac{V_{R2}}{R_2}$$

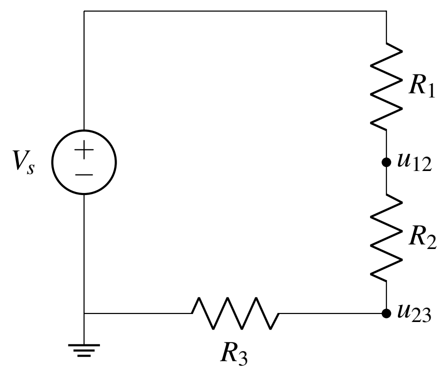
Writing the element voltages in terms of the node voltages, we have that $V_{R1} = V_s - u_{12}$ and $V_{R2} = u_{12}$.

Step 7: Substitute element currents into KCL Equations
Substituting back, we obtain $\frac{V_s - u_{12}}{R_1} = \frac{u_{12}}{R_2}$. Solving, we find that $u_{12} = \frac{R_2}{R_1+R_2} V_s$. Plugging in $u_{12} = 6V$ and $V_s = 12V$, we see that $R_1 = R_2$ must be true.

To choose R_1 and R_2 such that the current through each element is ≤ 1A, use KVL to write an expression for I_1, I_2 as a function of R_1, R_2:

$V_s - I_1 R_1 - I_2 R_2 = 0$, with $I_1 = I_2 = I_s$
$V_s = I_s(R_1 + R_2)$
$12V = 0.8A(R_1 + R_2)$
$R_1 + R_2 = 15\,\Omega$
As $R_1 + R_2$ must be at least $15\,\Omega$, and $R_1 = R_2$, we choose $R_1 = R_2 = 7.5\,\Omega$.

(b) Select values for R_1, R_2, R_3 in the circuit below such that $u_{12} = 6V$ and $u_{23} = 2V$.

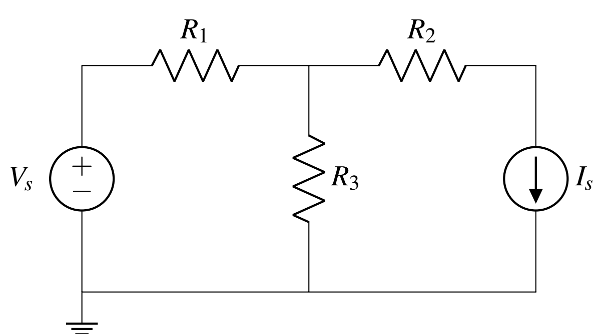


Solution: Let V_{R1}, V_{R2}, V_{R3} be the voltage drop across R_1, R_2, R_3 respectively. From KCL, we know that the current flowing through each of the elements is the same. Using Ohm's law, this can be rewritten as $\frac{V_{R1}}{R_1} = \frac{V_{R2}}{R_2} = \frac{V_{R3}}{R_3}$. Writing the element voltages in terms of the node voltages, we have that $V_{R1} = V_s - u_{12}$ and $V_{R2} = u_{12} - u_{23}$ and $V_{R3} = u_{23}$. Solving, similarly to the previous subpart, we find that $u_{12} = \frac{R_2+R_3}{R_1+R_2+R_3} V_s$ and $u_{23} = \frac{R_3}{R_1+R_2+R_3} V_s$. We found in the previous subpart that the total resistance in our KVL loop must be at least $15\,\Omega$ to limit the current to ≤ 0.8A, meaning for this circuit that $R_1 + R_2 + R_3 \geq 15\,\Omega$. Choosing $R_1 = 7.5\,\Omega, R_2 = 5\,\Omega, R_3 = 2.5\,\Omega$ would give us the desired voltage, and limit the current to ≤ 0.8A, though scaling the resistor values up by the same multiplicative constant would also be a valid solution.

2. Circuit Analysis

Using the steps outlined in lecture, analyze the following circuits to calculate the currents through each branch and the voltages at each node. Use the ground node labelled for you. You may use a numerical tool, such as IPython.

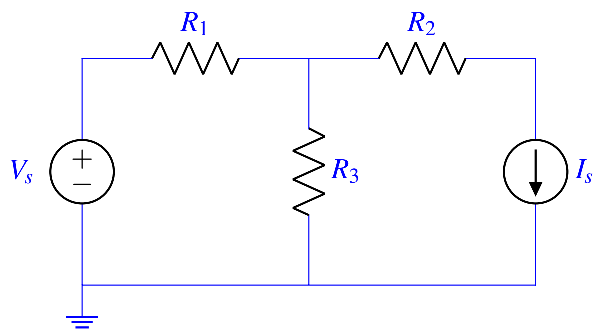
(a) $V_s = 5\,\mathrm{V}, I_s = 2\,\mathrm{A}, R_1 = R_2 = 2\,\Omega, R_3 = 4\,\Omega$



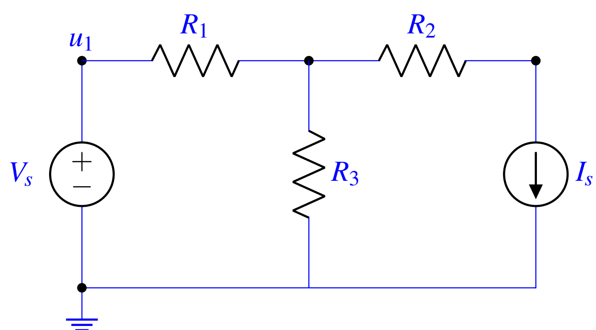
Solution:

Step 1) Reference node

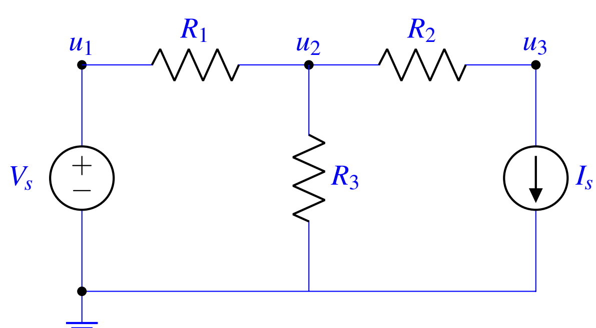
Select a reference (ground) node. Any node can be chosen for this purpose. In this example, we choose the node at the bottom of the circuit diagram.



Step 2) Label Nodes with Voltage set by Sources
Voltage sources set the voltage of the node they are connected to. In the example, there is only one source, V_s, and we label the corresponding source u_1 (names are arbitrary, but must be unique).

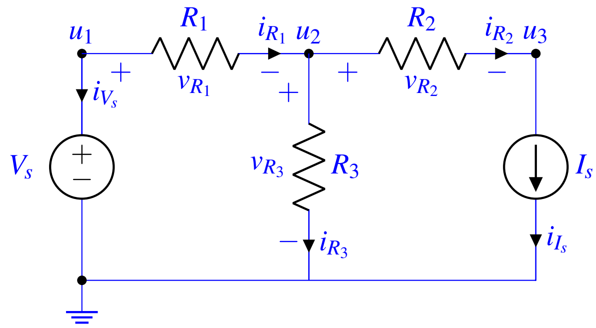


Step 3) Label Remaining Nodes
Now we label all remaining nodes in the circuit except the reference. In the example there are two, u_2 and u_3.



Step 4) Label Element Voltages and Currents

Next we mark all element voltages and currents.
Start with the current. The direction is arbitrary (top to bottom, bottom to top, it won't matter, but stick with your choice in subsequent steps). Then mark the element voltages following the passive sign convention, i.e. the voltage and current point in the "same" direction.



Step 5) KCL Equations
Write KCL equations for all nodes with unknown voltage, u_2 and u_3.

$$-i_{R_1} + i_{R_2} + i_{R_3} = 0$$
$$-i_{R_2} + i_{I_s} = 0$$

Step 6) Element Currents
Find expressions for all element currents in terms of voltage and element characteristics (e.g. Ohm's law) for all circuit elements except voltage sources. In this circuit there are four, R_1, R_2, R_3 and I_s.

$$i_{I_s} = I_s$$
$$i_{R_1} = \frac{V_{R_1}}{R_1}$$
$$i_{R_2} = \frac{V_{R_2}}{R_2}$$
$$i_{R_3} = \frac{V_{R_3}}{R_3}$$

We also have

$$u_1 = V_s$$
$$i_{I_s} = I_s$$
$$i_{R_1} = \frac{u_1 - u_2}{R_1}$$
$$i_{R_2} = \frac{u_2 - u_3}{R_2}$$
$$i_{R_3} = \frac{u_2}{R_3}$$

Step 7) Substitute Element Currents in KCL Equations
Now we substitute the expressions derived in Step 6 into the KCL equations from Step 5.

$$-\frac{V_s - u_2}{R_1} + \frac{u_2 - u_3}{R_2} + \frac{u_2}{R_3} = 0$$
$$-\frac{u_2 - u_3}{R_2} + I_s = 0$$

Let's make this a bit nicer by grouping the unknowns (u_2 and u_3) on the left side and the known terms on the right:

$$u_2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) + u_3\left(-\frac{1}{R_2}\right) = \frac{V_s}{R_1}$$
$$u_2\left(-\frac{1}{R_2}\right) + u_3\left(\frac{1}{R_2}\right) = -I_s$$

Note that we now have 2 equations for 2 unknowns. Thus, we set up the following matrix relation:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{V_s}{R_1} \\ -I_s \end{bmatrix}$$

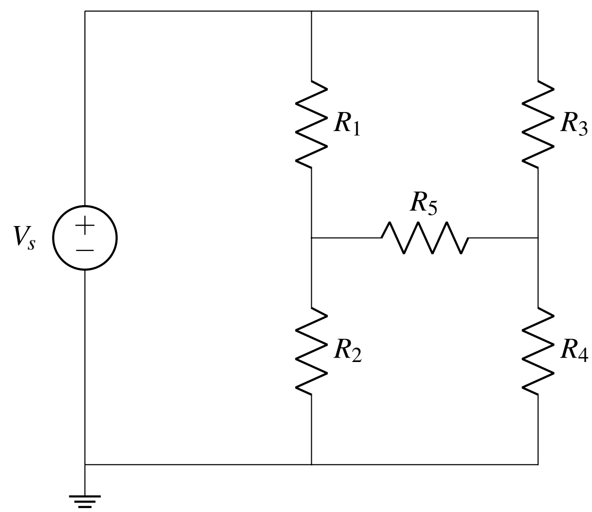
Finally, we plug in the values we were given into the matrix above and use Gaussian elimination to find the vector of unknowns. We find that:

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.667V \\ -3.33V \end{bmatrix}$$

The branch currents can be obtained from the node voltages and element equations. Therefore, we can write:

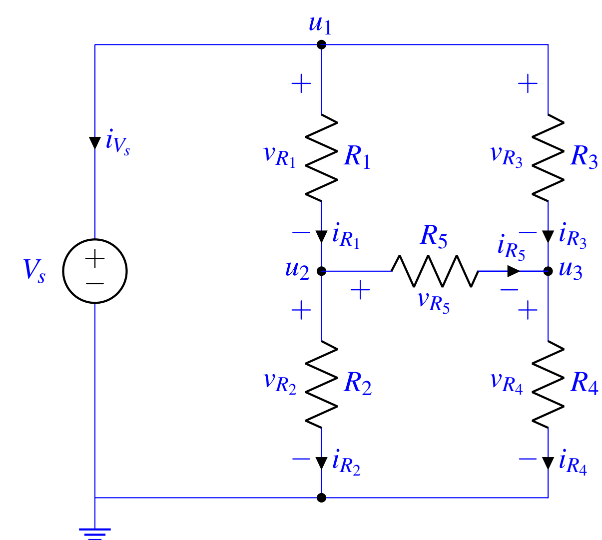
$$i_{V_s} = -2.167A$$
$$i_{I_s} = I_s = 2A$$
$$i_{R_1} = \frac{u_1 - u_2}{R_1} = 2.167A$$
$$i_{R_2} = \frac{u_2 - u_3}{R_2} = 2A$$
$$i_{R_3} = \frac{u_2}{R_3} = 0.167A$$

(b) $V_s = 5\,\mathrm{V}, R_1 = 1\,\Omega, R_2 = 2\,\Omega, R_3 = 3\,\Omega, R_4 = 4\,\Omega, R_5 = 5\,\Omega$



Solution:

Here, we will skip showing all of the individual steps. Below is the circuit with our choice of ground and current directions.



From the above circuit, we get the following KCL equations:

$$-i_{R_1} + i_{R_2} + i_{R_5} = 0$$
$$-i_{R_3} + i_{R_4} - i_{R_5} = 0$$

Using the IV relations for each element, we have:

$$
\begin{aligned}
u_1 - 0 &= V_s \\
i_{R_1} &= \frac{u_1 - u_2}{R_1} \\
i_{R_2} &= \frac{u_2 - 0}{R_2} \\
i_{R_3} &= \frac{u_1 - u_3}{R_3} \\
i_{R_4} &= \frac{u_3}{R_4} \\
i_{R_5} &= \frac{u_2 - u_3}{R_5}
\end{aligned}
$$

We also know that $u_1 = V_S$

Now we substitute these expressions into the KCL equations we previously derived.

$$
\begin{aligned}
-\frac{V_s - u_2}{R_1} + \frac{u_2 - 0}{R_2} + \frac{u_2 - u_3}{R_5} &= 0 \\
-\frac{V_s - u_3}{R_3} + \frac{u_3}{R_4} - \frac{u_2 - u_3}{R_5} &= 0
\end{aligned}
$$

Let's make this a bit nicer by grouping the unknowns (u_2 and u_3) on the left side and the known terms on the right:

$$
\begin{aligned}
u_2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_5}\right) + u_3\left(-\frac{1}{R_5}\right) &= \frac{V_s}{R_1} \\
u_2\left(-\frac{1}{R_5}\right) + u_3\left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right) &= \frac{V_s}{R_3}
\end{aligned}
$$

Note that we now have 2 equations for 2 unknowns. Thus, we set up the following matrix relation:

$$
\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_5} & -\frac{1}{R_5} \\ -\frac{1}{R_5} & \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{V_s}{R_1} \\ \frac{V_s}{R_3} \end{bmatrix}
$$

Finally, we plug in the values we were given into the matrix above and use Gaussian elimination to find the vector of unknowns.

$$
\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 3.29V \\ 2.968V \end{bmatrix}
$$

The branch currents can be obtained from the node voltages and element equations. Therefore, we can write:

$$i_{R_1} = \frac{u_1 - u_2}{R_1} = 1.709A$$
$$i_{R_2} = \frac{u_2 - 0}{R_2} = 1.645A$$
$$i_{R_3} = \frac{u_1 - u_3}{R_3} = 0.677A$$
$$i_{R_4} = \frac{u_3}{R_4} = 0.741A$$
$$i_{R_5} = \frac{u_2 - u_3}{R_5} = 0.0644A$$
$$i_{V_s} = i_{R_1} + i_{R_3} = -2.38A$$

3. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.