EECS 16A Designing Information Devices and Systems I
Summer 2020 Homework 2A

This homework is due Wednesday July 8, 2020, at 23:59 PT.
Self-grades are due Sunday July 12, 2020, at 23:59 PT.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned). This homework does not contain any IPython components.

Homework Learning Goals: *You should get practice with computing nullspaces and columnspaces, which is then applicable to computing eigenvalues and eigenvectors. Each of these concepts provides an important characterization of matrices.*

1. Finding Null Spaces and Column Spaces

Learning Objectives*: Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $A \in \mathbb{R}^{m\times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. The null space is notated as $N(A)$ and the definition can be written in set notation as:

$$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $A \in \mathbb{R}^{m\times n}$, is the set of all vectors $A\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of A's columns. The column space can be notated as $\mathrm{C}(A)$ or $\mathrm{range}(A)$ and the definition can be written in set notation as:

$$\mathrm{C}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

(a) Consider matrices in $\mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors?

Solution: If you are stuck solving a problem like this, consider concrete examples. We want to find the maximum possible number of linearly independent column vectors, so we look for examples and check if we can exceed certain values.

Consider the following example matrix, where the entries marked with $*$ are arbitrary values:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

The first three columns are linearly independent, so at least three linearly independent columns are achievable. The first three columns span $\mathbb{R}^3$, therefore any choice of fourth and fifth columns, also in $\mathbb{R}^3$, can be written as a linear combination of the first three columns. This means that we cannot exceed three linearly independent columns. Thus the maximum number of linearly independent column vectors is 3. In general, the columns of $\mathbf{A} \in \mathbb{R}^{m\times n}$ where $m < n$ will always be linearly dependent.

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a set of vectors that span the column space of $\mathbf{A}$. What is the minimum number of vectors required to span the column space of $\mathbf{A}$? (This is the dimension of the column space of $\mathbf{A}$.)

Solution: The column space of $\mathbf{A}$ is the space spanned by its columns, so the set of all columns is a valid answer. However, it is possible to choose a subset of the columns and still span the column space, as we will be guaranteed linear dependence of the columns, as seen in part (a). To find the minimum number of columns needed, we can remove vectors from our set until we are left with a linearly independent set.

By inspection, the second, fourth, and fifth columns can be omitted from a set of columns as they can be expressed as linear combinations of the first and third columns. Thus the minimum number of vectors required to span the column space is 2.

One set spanning the range of $\mathbf{A}$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Another valid set of vectors which span the column space is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Note with this second set, none of the columns of $\mathbf{A}$ appear. Despite this, the span of this set will still be equal to the column space which is the set of all vectors in $\mathbb{R}^3$ with zero third entry. Give yourself full credit if you recognized that the minimum number of vectors required was 2, and if you had any set that spans the column space.

(c) The dimension of the null space is the minimum number of vectors needed to span it. Find a set of vectors that span the null space of $\mathbf{A}$ (the matrix from part (b)). What is the dimension of the null space of $\mathbf{A}$?

Solution:

Finding the null space of $\mathbf{A}$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We observe that x_2, x_4, and x_5 are free variables. Thus, we let x_2, x_4 and x_5 be a, b and c. Now we rewrite the equations as:

$$\begin{aligned} x_1 &= -a + 2b - 3c \\ x_2 &= a \\ x_3 &= b - c \\ x_4 &= b \\ x_5 &= c \end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of $\mathbf{A}$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of the null space is 3, as it is the minimum number of vectors we need to span it.

(d) What do you notice about the sum of the dimensions of the null space and the column space in relation to the dimensions of $\mathbf{A}$?

Solution: The dimensions of the column space and the null space add up to the number of columns in A. This is true of all matrices and is called the Rank-Nullity Theorem.

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a set of vectors that span the column space of $\mathbf{B}$. What is the minimum number of vectors required to span the column space of $\mathbf{B}$?

Solution:

We see that only the first two column vectors of B span the column space of B. The third column does not contribute to the column space of B. There are only two unique vectors in the set and therefore the column space has dimension 2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(f) Find a set of vectors that spans the null space of the following matrix. This problem requires systematic calculations, but is helpful if you want more practice.

$$\mathbf{C} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

Solution: To find the null space, we wish to solve for all $\vec{x}$ such that $A\vec{x} = \vec{0}$. However, as the right hand side of the augmented matrix will always remain a column of zeroes under any row operation, we can omit the column and row reduce just A.

$$\begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 & 4 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \qquad \frac{1}{2}R_1 \rightarrow R_1$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \qquad \begin{matrix} R_1 - R_2 \rightarrow R_2 \\ 2R_1 - R_3 \rightarrow R_3 \\ 3R_1 - R_4 \rightarrow R_4 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} 2R_2 + R_1 \rightarrow R_1 \\ R_2 - R_3 \rightarrow R_3 \\ R_2 - R_4 \rightarrow R_4 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad -R_2 \rightarrow R_2$$

Vectors in the null space satisfy the following equations:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} \implies \begin{aligned} x_1 - 2x_2 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

We then set x_2 and x_4 to be the free variables a and b respectively and substitute in:

$$\begin{aligned} x_1 &= 2a \\ x_2 &= a \\ x_3 &= -2b \\ x_4 &= b \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the null space of the matrix is spanned by the vectors:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(g) Find the column space and its dimension, and the nullspace and its dimension of the following matrix.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find the null space and column space, we can row reduce the matrix to find solutions to $D\vec{x} = \vec{0}$ which are in $\mathrm{N}(D)$, and also identify which vectors can be written as a linear combination of the others and should therefore be discarded from a basis for $\mathrm{C}(D)$. We do not need an augmented column of $\vec{0}$ as it will not change under row operations.

$$\begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix} \overrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we only have pivots in the first and third columns, we can assign the free variables x_2 and x_4 to s and t. We can write all solutions to $D\vec{x} = \vec{0}$ as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

Our vectors in $\mathrm{N}(D)$ can be written as a linear combination of two vectors: $\mathrm{N}(D) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Since the vectors we are taking a span of are also linearly independent, we have a basis, and so the dimension of the null space is 2.

Our pivots in the first and third columns tell us that we only need to take the corresponding columns to span the column space. $\mathrm{C}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \end{bmatrix} \right\}$. We are also taking a span of linearly independent vectors here, so we have a basis. Since the basis has two elements, the dimension of the column space is 2.

2. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice algorithmic computation of eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this my inspection if you prefer.

(a) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix}0\\1\end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix}1\\0\end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\det\left(\begin{bmatrix}5-\lambda & 0\\ 0 & 2-\lambda\end{bmatrix}\right)=0$$
$$(5-\lambda)(2-\lambda)-0=0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda=5$:

$$\mathbf{A}\vec{x}=5\vec{x}\implies(\mathbf{A}-5\mathbf{I}_2)\vec{x}=\vec{0}$$
$$\left(\begin{bmatrix}5 & 0\\ 0 & 2\end{bmatrix}-\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\implies\begin{bmatrix}0 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\implies y=0\implies\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix}1\\0\end{bmatrix}\}$s an eigenvector with corresponding eigenvalue $\lambda=5$ is .

$\lambda=2$:

$$\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\implies x=0\implies\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\y\end{bmatrix}$$

where y is a free variable.

Any vector in span$\{\begin{bmatrix}0\\1\end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda=2$.

(b) $\begin{bmatrix}22 & 6\\ 6 & 13\end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$\det\left(\begin{bmatrix}22-\lambda & 6\\ 6 & 13-\lambda\end{bmatrix}\right)=0$$
$$(22-\lambda)(13-\lambda)-36=0\implies\lambda=10,25$$

$\lambda=10$:

$$\begin{bmatrix}12 & 6\\ 6 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\implies\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\implies 2x+y=0\implies\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\-2x\end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix}1\\-2\end{bmatrix}\}$ is an eigenvector with corresponding eigenvalue $\lambda=10$.

$\lambda=25$:

$$\begin{bmatrix}-3 & 6\\ 6 & -12\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\implies\begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\implies 2y=x\implies\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2y\\y\end{bmatrix}$$

where y is a free variable.

Any vector that lies in span$\{\begin{bmatrix}2\\1\end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda=25$.

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Solution:

This can also be seen by inspection. The matrix is not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

Alternatively, we can explicitly calculate.

$$\det(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}) = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$
$$\lambda^2 - 5\lambda = 0 \implies \lambda(\lambda - 5) = 0$$
$$\lambda = 0, 5$$

$\lambda = 0$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$x + 2y = 0 \implies y = -\frac{1}{2}x \implies \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Any vector that lies in span$\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.

$\lambda = 5$:

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$2x - y = 0 \implies y = 2x \implies \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Any vector that lies in span$\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 5$.

(d) Let $A \in \mathbb{R}^{n \times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of this matrix

$$\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace.

Solution:

The zero vector is contained in this set since $A\vec{0} = \vec{0} = \lambda\vec{0}$.

Let $\vec{v}_1$ and $\vec{v}_2$ be members of the set. Let $\vec{u} = \alpha\vec{v}_1$. Now, $A\vec{u} = A\alpha\vec{v}_1 = \alpha A\vec{v}_1 = \alpha\lambda\vec{v}_1 = \lambda\vec{u}$. Hence, $\vec{u}$ is a member of the set as well and the set is closed under scalar multiplication.

Observe below that the set is closed under vector addition as well.

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$$

Hence, the set defined in the question satisfies the properties of a subspace and is consequently a subspace of $\mathbb{R}^n$.

3. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.