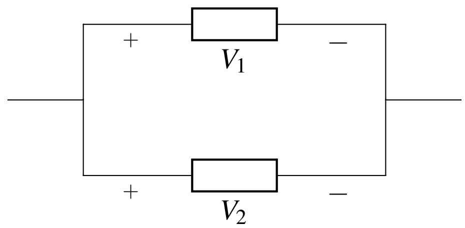
EECS 16A Designing Information Devices and Systems I
Summer 2020 Module 2 Practice Handout

1. Circuits Intuition Practice

(a) What does KVL tell you about V_1 and V_2 for any elements connected to the same pair of nodes?

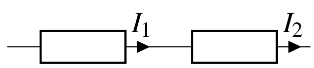


Solution: Going around the loop starting from the left node, and using KVL, we can write:

$$V_1 - V_2 = 0$$

The 2 elements are connected in **parallel**. This means they will have the same **voltage** across them.

(b) What does KCL tell you about I_1 and I_2 for any two elements connected to a node with nothing else connected to that node?

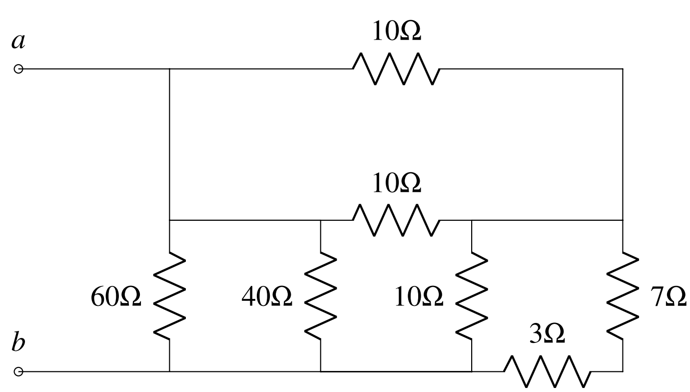


Solution: Using KCL at the center node the relationship for the currents are:

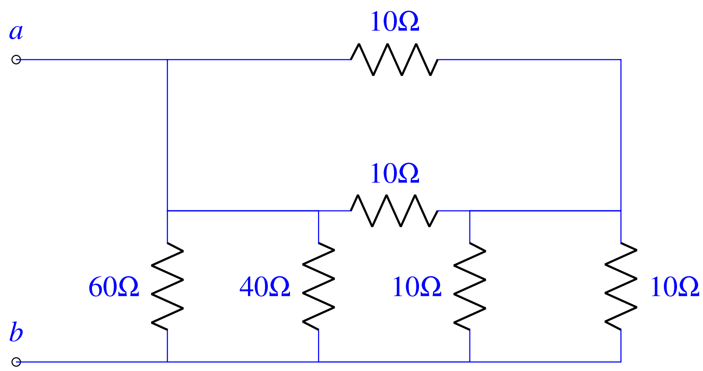
$$I_1 = I_2$$

For any two elements that are connected to a node with nothing else connected to them, the current values will be identical. The 2 elements are connected in **series**.

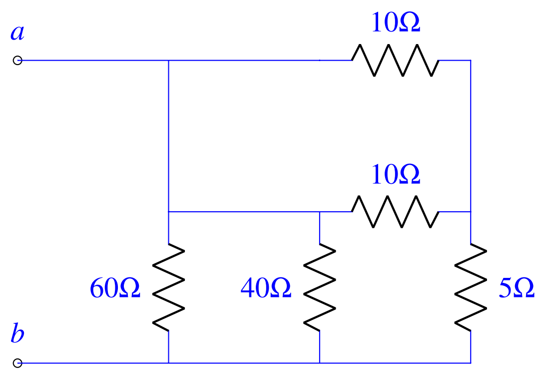
(c) Find R_{ab}, the equivalent resistance between terminals a and b. Give your answer as a number, or an expression involving no more than one use of $||$.



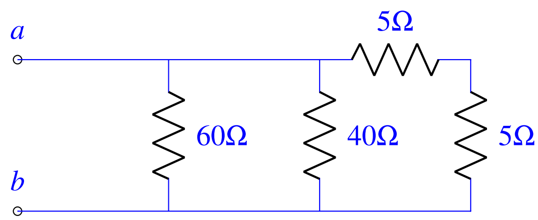
Solution: The 3Ω and 7Ω resistors are in series. Combining them, we get 10Ω.



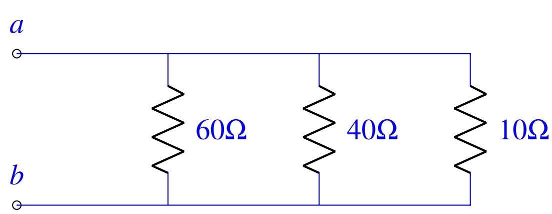
The two lower resistors with values of 10Ω are in parallel. Combining them, we get $10||10\Omega = 5\Omega$.



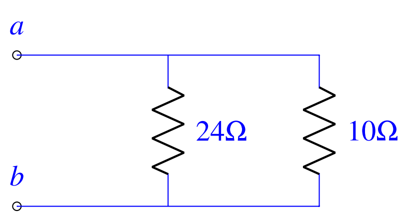
The upper two resistors with values of 10Ω are also in parallel. We get another 5Ω resistor.



Now, we can combine the two 5Ω resistors into one 10Ω resistor.

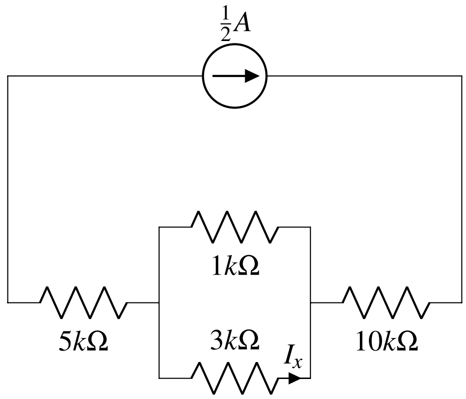


The equivalent of the 60Ω and 40Ω resistors in parallel are $R_{eq} = \frac{60\Omega \cdot 40\Omega}{60\Omega + 40\Omega} = 24\Omega$.

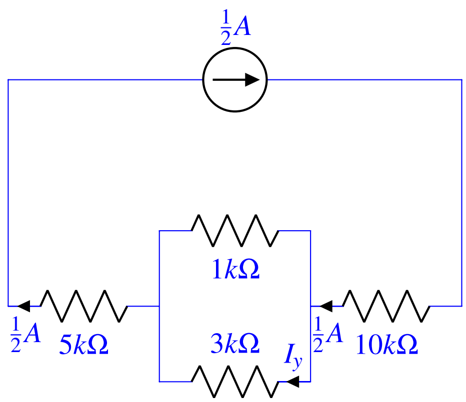


The final equivalent resistance R_{ab} can be written as $R_{ab} = 24\Omega||10\Omega$, or as $R_{ab} = \frac{24 \cdot 10}{24+10}\Omega = \frac{120}{17}\Omega \approx 7.058\Omega$.

(d) Find I_x. (Hint: Can you see the current divider?)



Solution: By KCL at the right node of $10k\Omega$, the current that goes through $10k\Omega$ is the same as the current that comes from the source. By KCL at the left node of $10k\Omega$, the current that goes through $10k\Omega$ is the same as the sum of currents that go through $1k\Omega$ and $3k\Omega$. We can consider the resistors $1k\Omega$ and $3k\Omega$ as a current divider, since the current that goes through $5k\Omega$ is also $\frac{1}{2}A$.



The current I_y will therefore be:

$$I_y = \frac{1k\Omega}{1k\Omega + 3k\Omega}\frac{1}{2}A = \frac{1}{4} \cdot \frac{1}{2}A = \frac{1}{8}A$$

To be consistent with the original current direction labeled, $I_x = -I_y = -\frac{1}{8}A$

2. Digital to Analog Converter (DAC)

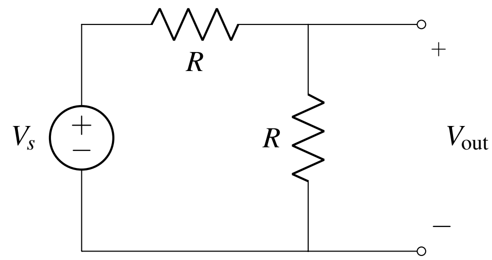
For some outputs, such as audio applications, we need to produce an analog output, or a continuous voltage from 0 to V_s. These analog voltages must be produced from digital voltages, that is sources, that can only be V_s or 0. A circuit that does this is known as a Digital to Analog Converter. It takes a binary representation of a number and turns it into an analog voltage.

The output of a DAC can be represented with the equation shown below:

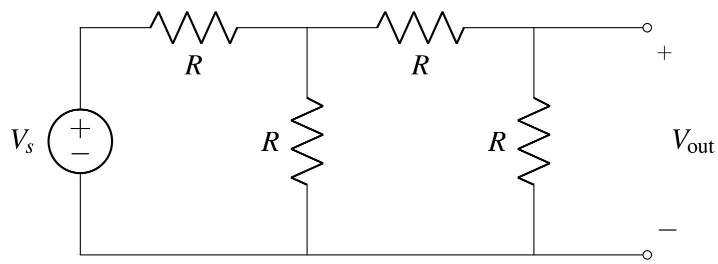
$$V_{\text{out}} = V_s \sum_{n=0}^{N} \frac{1}{2^n} \cdot b_n$$

where each binary digit b_n is multiplied by $\frac{1}{2^n}$.

(a) We know how to take an input voltage and divide it by 2:



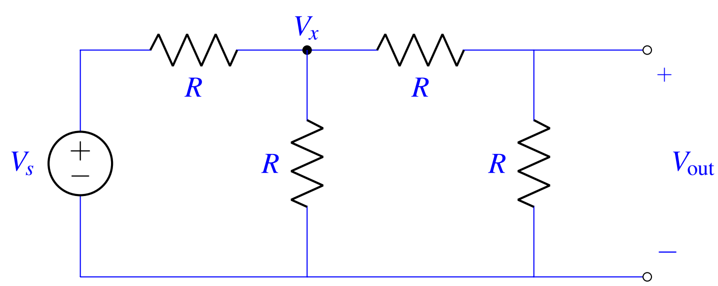
To divide by larger powers of two, we might hope to just "cascade" the above voltage divider. For example, consider:



Calculate V_{out} in the above circut. Is $V_{\text{out}} = \frac{1}{4}V_s$?

Solution:

We first find the potential V_x.



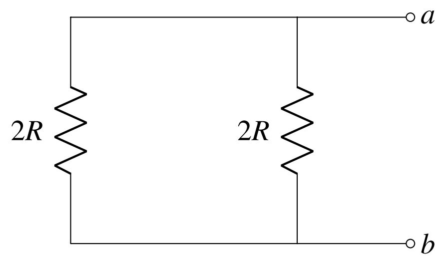
$$V_x = \frac{R \parallel 2R}{R + R \parallel 2R} V_s = \frac{\frac{2}{3}R}{R + \frac{2}{3}R} V_s = \frac{2}{5} V_s$$

$$V_{\text{out}} = \frac{R}{R+R} V_x = \frac{1}{2} \cdot \frac{2}{5} V_s = \frac{1}{5} V_s \neq \frac{1}{4} V_s$$

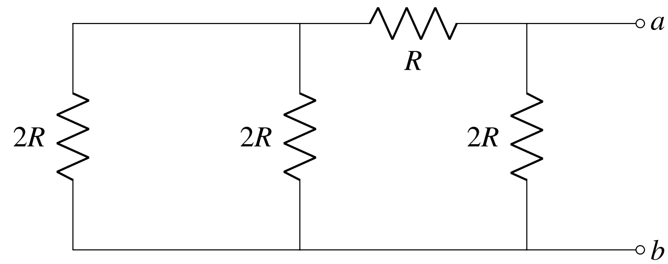
No, V_{out} does not equal $\frac{1}{4}V_s$.

(b) The R-$2R$ ladder, shown below, has a very nice property. For each of the circuits shown below, find the equivalent resistance looking in from points a and b. Do you see a pattern?

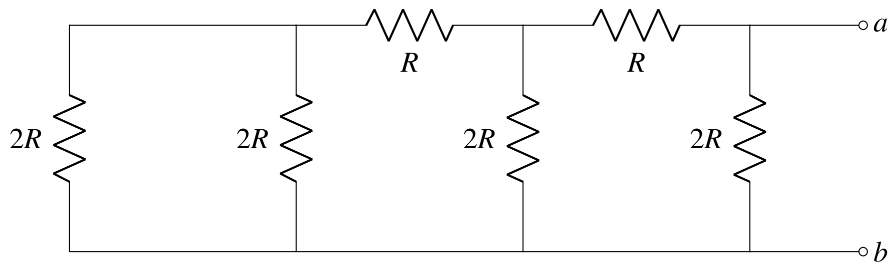
i.



ii.



iii.

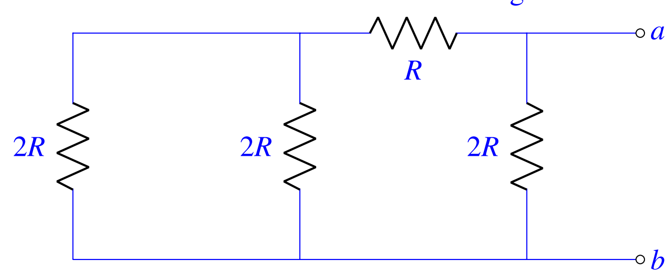


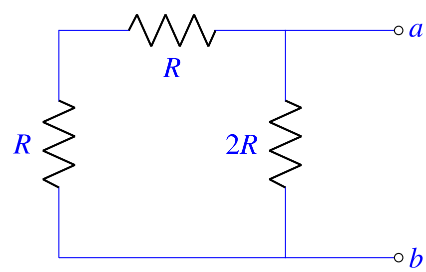
Solution:

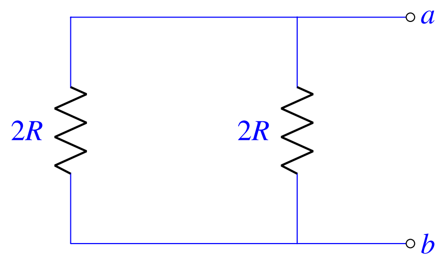
i.

$$R_{eq} = 2R \parallel 2R = R$$

ii. We find the equivalent resistance for the resistors from left to right.

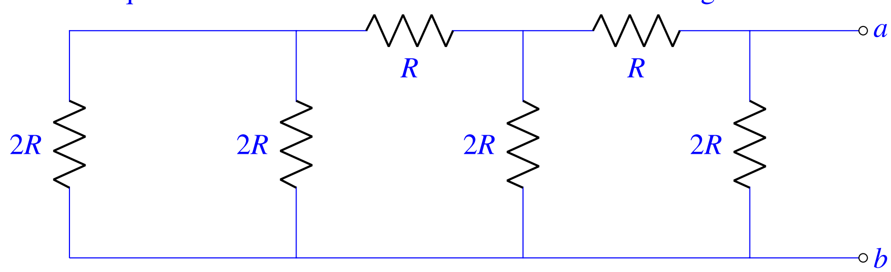


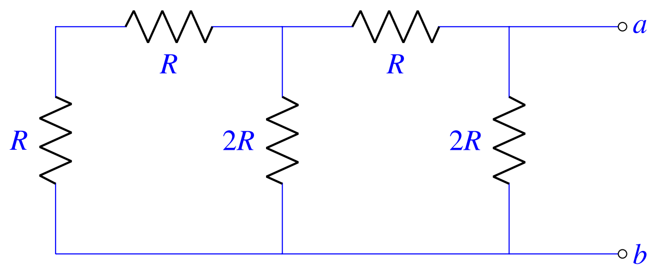


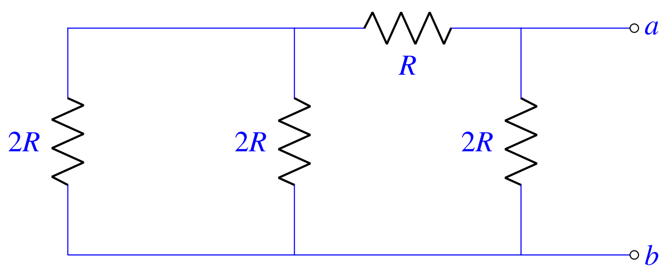


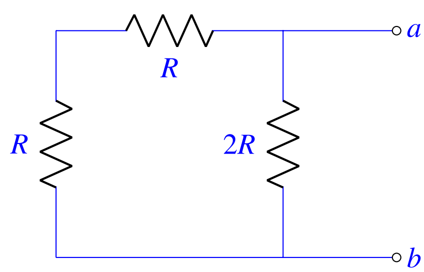
$$R_{eq} = 2R \parallel 2R = R$$

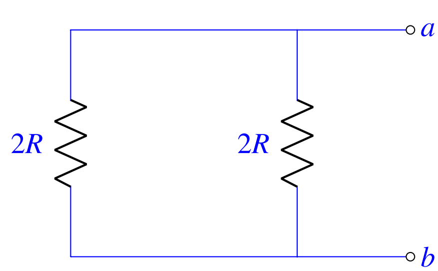
iii. Again, we find the equivalent resistance for the resistors from left to right.







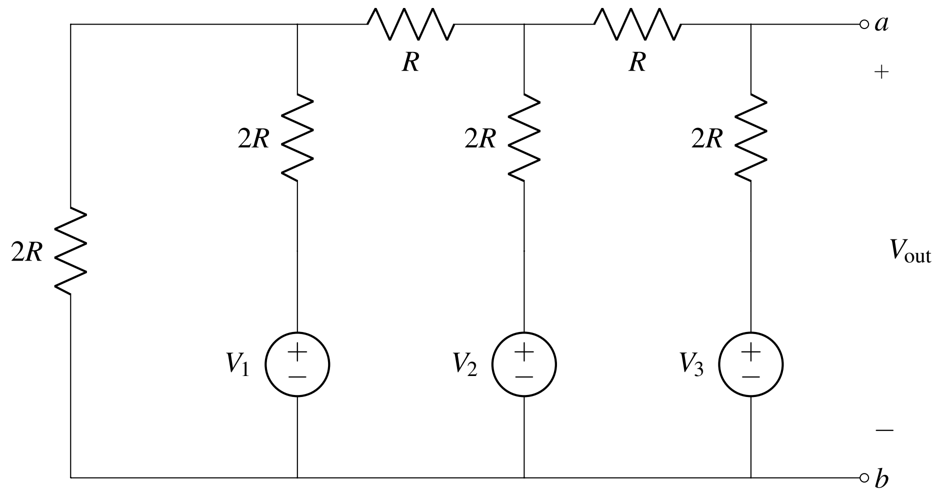




$$R_{eq} = 2R \parallel 2R = R$$

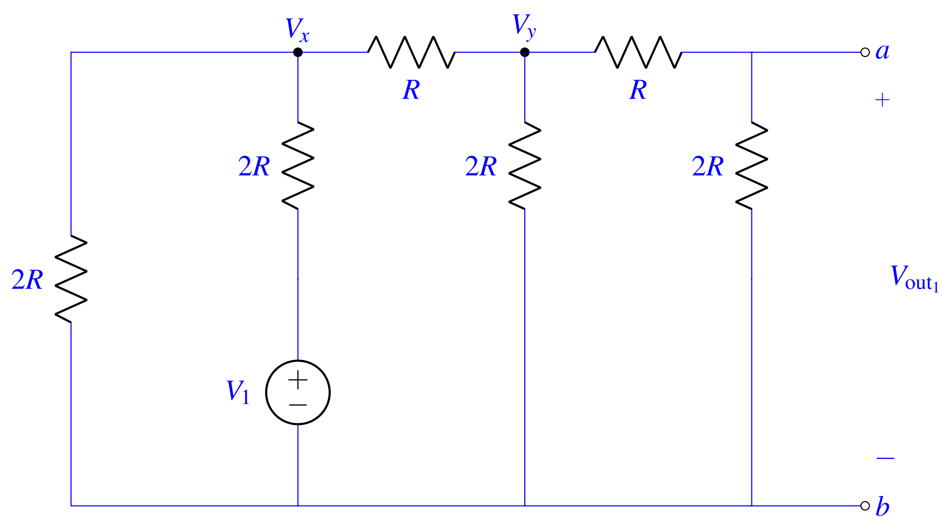
The equivalent resistance is always $R_{eq} = R$.

(c) The following circuit is an R-$2R$ DAC. To understand its functionality, use superposition to find V_{out} in terms of each V_k in the circuit.

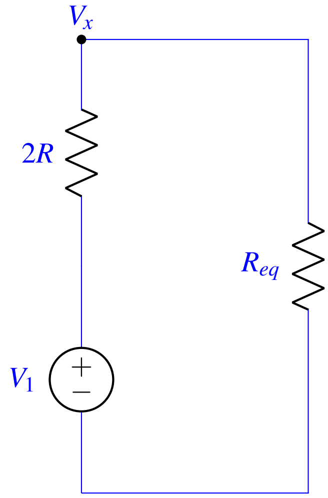


Solution:

V_1:



We first find the potential V_x. To do this, we can simplify the circuit.



$$R_{eq} = 2R \parallel (R + (2R \parallel (R + 2R))) = \frac{22}{21}R$$

We can then find V_x using the voltage divider formula.

$$V_x = \frac{\frac{22}{21}R}{2R + \frac{22}{21}R}V_1 = \frac{11}{32}V_1$$

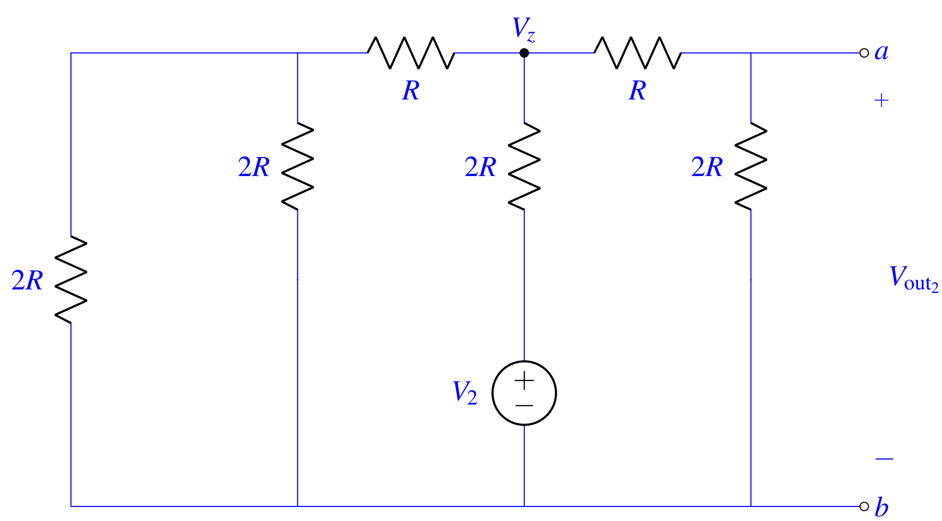
Similarly, we use the voltage divider formula to find V_y in terms of V_x.

$$V_y = \frac{2R \parallel (R + 2R)}{R + 2R \parallel (R + 2R)}V_x = \frac{\frac{6}{5}R}{R + \frac{6}{5}R}V_x = \frac{6}{11} \cdot \frac{11}{32}V_1 = \frac{3}{16}V_1$$

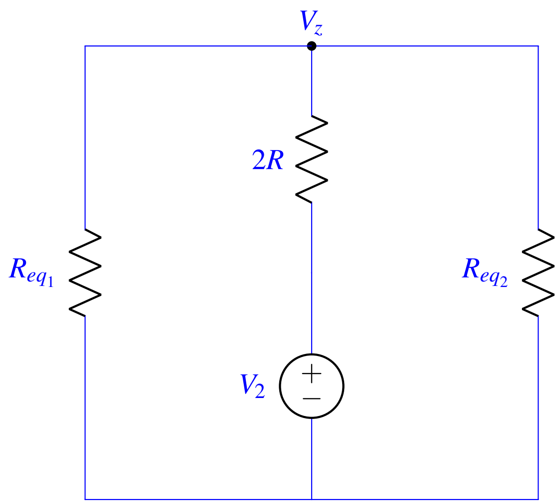
Applying the voltage divider formula again gives us V_{out_1}.

$$V_{\text{out}_1} = \frac{2R}{R + 2R}V_y = \frac{2}{3} \cdot \frac{3}{16}V_1 = \frac{1}{8}V_1$$

V_2:



We first find the potential V_z. To do this, we can simplify the circuit.



$$R_{eq_1} = R + (2R \parallel 2R) = R + R = 2R$$

$$R_{eq_2} = R + 2R = 3R$$

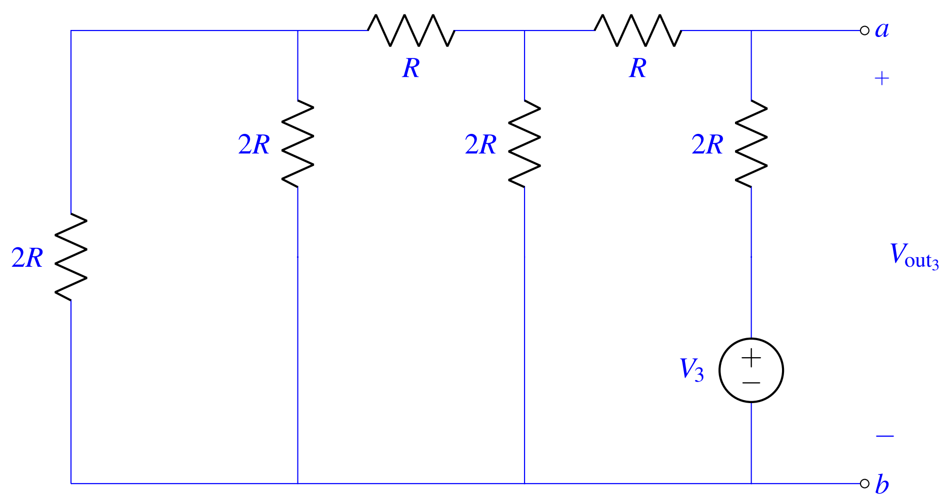
We can then find V_z using the voltage divider formula.

$$V_z = \frac{2R \parallel 3R}{2R + (2R \parallel 3R)} V_2 = \frac{\frac{6}{5}R}{2R + \frac{6}{5}R} V_2 = \frac{3}{8} V_2$$

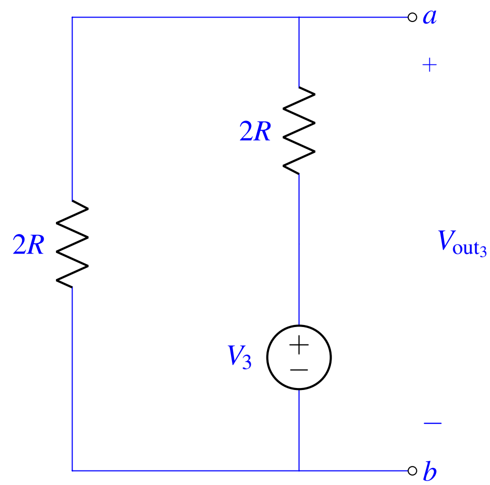
Applying the voltage divider formula again gives us V_{out_2}.

$$V_{\text{out}_2} = \frac{2R}{R + 2R} V_z = \frac{2}{3} \cdot \frac{3}{8} V_2 = \frac{1}{4} V_2$$

V_3:



We can simplify this circuit.



$$V_{out_3} = \frac{2R}{2R+2R} V_3 = \frac{1}{2} V_3$$

$$V_{out} = V_{out_1} + V_{out_2} + V_{out_3} = \frac{1}{8} V_1 + \frac{1}{4} V_2 + \frac{1}{2} V_3$$

(d) We've now designed a 3-bit R-$2R$ DAC. What is the output voltage V_{out} if $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$?

Solution:

$$V_{out} = \frac{1}{8} \cdot 0\,\text{V} + \frac{1}{4} \cdot 1\,\text{V} + \frac{1}{2} \cdot 0\,\text{V} = 1/4\,\text{V}$$

3. Measuring Voltage and Current

In order to measure quantities such as voltage and current, engineers use voltmeters and ammeters. A simple model of a voltmeter is a resistor with a very high resistance, R_{VM}. **The voltmeter measures the voltage across the resistance** R_{VM}. The measured voltage is then relayed to a microprocessor (such as the MSP430s used in Lab).

This model of an voltmeter is shown in Figure 1. Let us explore what happens when we connect this voltmeter to various circuits to measure voltages.
Throughout this problem assume $R_{VM} = 1M\Omega$. Recall that the SI prefix M or Mega is 10^6.

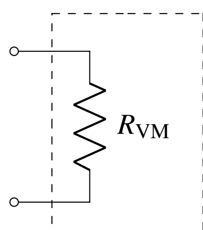


Figure 1: Our model of a voltmeter, $R_{VM} = 1M\Omega$

(a) Suppose we wanted to measure the voltage across R_2 (v_{out}) produced by the voltage divider circuit shown in Figure 2 on the left. The circuit on the right in Figure 2 shows how we would connect the

voltmeter across R_2. Assume $R_1 = 100\Omega$ and $R_2 = 200\Omega$.
First calculate the value of v_{out}. Then calculate the voltage the voltmeter would measure, i.e. v_{meas}.

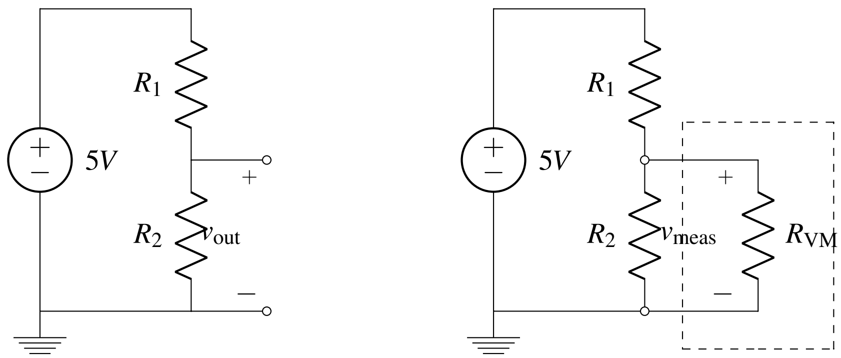


Figure 2: Left: Circuit without the voltmeter connected, Right: Voltmeter measuring voltage across R_2

Solution: We start by finding v_{out} in the circuit on the left. Recognizing that this circuit is a voltage divider, we can directly find the following:

$$v_{\text{out}} = \frac{R_2}{R_1 + R_2} 5\text{V} = \frac{200\,\Omega}{300\,\Omega} 5\text{V} = 3.3333\text{V}$$

Next we consider the circuit on the right. We start by combining the resistor R_2 and R_{VM} since they are in parallel. Then we can apply the voltage divider formula to calculate the voltage across R_{VM}.

$$R_2||R_{\text{VM}} = \frac{R_2 R_{\text{VM}}}{R_2 + R_{\text{VM}}} = \frac{1\,\text{M}\Omega \cdot 200\,\Omega}{1\,\text{M}\Omega + 200\,\Omega} = 199.96\,\Omega$$

$$v_{\text{out}} = \frac{R_2||R_{\text{VM}}}{R_1 + R_2||R_{\text{VM}}} \cdot 5\,\text{V} = \frac{199.96\,\Omega}{199.96\,\Omega + 100\,\Omega} \cdot 5\,\text{V} = 3.3331\,\text{V}$$

(b) Repeat part (a), but now $R_1 = 10M\Omega$ and $R_2 = 10M\Omega$. Is this voltmeter still a good tool to measure the output voltage?

Solution: We start by again finding v_{out} in the circuit on the left. Recognizing that this circuit is a voltage divider, we can directly find the following:

$$v_{\text{out}} = \frac{R_1}{R_1 + R_2} 5\text{V} = \frac{10\,\text{M}\Omega}{20\,\text{M}\Omega} 5\text{V} = 2.5\text{V}$$

Next we consider the circuit on the right. We start by combining the resistor R_2 and R_{VM} since they are in parallel. Then we can apply the voltage divider formula to calculate the voltage across R_{VM}.

$$R_2||R_{\text{VM}} = \frac{R_2 R_{\text{VM}}}{R_2 + R_{\text{VM}}} = \frac{10\,\text{M}\Omega \cdot 1\,\text{M}\Omega}{10\,\text{M}\Omega + 1\,\text{M}\Omega} = 0.909\,\text{M}\Omega$$

$$v_{\text{out}} = \frac{R_2||R_{\text{VM}}}{R_1 + R_2||R_{\text{VM}}} \cdot 5\,\text{V} = \frac{0.909\,\text{M}\Omega}{0.909\,\text{M}\Omega + 10\,\text{M}\Omega} \cdot 5\,\text{V} = 0.4166\,\text{V}$$

Since the resistors R_1 and R_2 are larger than R_{VM}, using the voltmeter to measure element voltages significantly changes the value of V_{out}. Thus our voltmeter is not a good tool to use to measure the voltage for this circuit.

(c) Now suppose we are working with the same circuit as in Part (a), but we know that $R_2 = R_1$. What is the maximum value of R_1 that ensures that the difference between voltage measurement of the voltmeter (v_{meas}) and the actual value (v_{out}) remains within $\pm 10\%$ of v_{out}?

Solution: We will only have to consider the case where v_{meas} is less than v_{out}, because the parallel combination of R_2 and R_{VM} can only make a resistor with total resistance smaller than R_2.

First, let's symbolically represent what the outputs are in the two cases:

For the circuit without the voltmeter connected:

$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} \cdot V_s$$

For the circuit with the voltmeter connected:

$$R_{\text{VM}} || R_2 = \frac{R_{\text{VM}} R_2}{R_{\text{VM}} + R_2}$$

$$V_{\text{meas}} = \frac{\frac{R_{\text{VM}} R_2}{R_{\text{VM}} + R_2}}{R_1 + \frac{R_{\text{VM}} R_2}{R_{\text{VM}} + R_2}} \cdot V_s = \frac{R_{\text{VM}} R_2}{R_1 (R_{\text{VM}} + R_2) + R_{\text{VM}} R_2} \cdot V_s$$

Now we need:

$$\frac{V_{\text{out}} - V_{\text{meas}}}{V_{\text{out}}} \leq \frac{1}{10}$$

$$\frac{\frac{R_2}{R_1 + R_2} \cdot V_s - \frac{R_{\text{VM}} R_2}{R_1 (R_{\text{VM}} + R_2) + R_{\text{VM}} R_2} \cdot V_s}{\frac{R_2}{R_1 + R_2} \cdot V_s} \leq \frac{1}{10}$$

$$\frac{\frac{R_2}{R_1 + R_2} - \frac{R_{\text{VM}} R_2}{R_1 (R_{\text{VM}} + R_2) + R_{\text{VM}} R_2}}{\frac{R_2}{R_1 + R_2}} \leq \frac{1}{10}$$

Since we know $R_1 = R_2$, we can simplify our final expression:

$$\frac{\frac{1}{2} - \frac{R_{\text{VM}}}{R_{\text{VM}} + R_2 + R_{\text{VM}}}}{\frac{1}{2}} \leq \frac{1}{10}$$

$$1 - \frac{2R_{\text{VM}}}{2R_{\text{VM}} + R_2} \leq \frac{1}{10}$$

$$\frac{R_2}{2R_{\text{VM}} + R_2} \leq \frac{1}{10}$$

$$\frac{9}{10} R_2 \leq \frac{2}{10} R_{\text{VM}}$$

$$R_2 \leq \frac{2}{9} R_{\text{VM}}$$

$$R_1 = R_2 \leq 0.22\,\text{M}\Omega$$

(d) Using the combination of our voltmeter and an additional resistor R_x, we can make an ammeter and measure the current through an element. Using the circuit shown in Figure 3, where $R_x = 1\,\Omega$, then the measured current through R_x is $I_{\text{meas}} = \frac{V_{\text{VM}}}{R_x}$ where V_{VM} is the voltage across the voltmeter.
In Figure 4, the voltmeter is connected to measure the current through resistor $R_1 = 1k\Omega$. For the circuit on the left, find the current through R_1 without the voltmeter connected (i.e. I_1). Then, for the circuit on the right, find the current measured by the voltmeter when it is connected as an ammeter (i.e. I_{meas}).

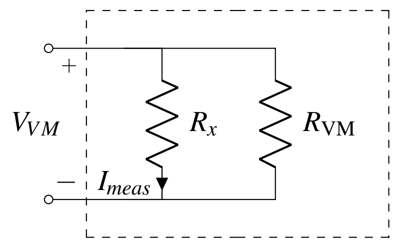


Figure 3: The voltmeter combined with resistor R_x to function as an ammeter (i.e. to measure current), $R_{\text{VM}} = 1M\Omega$.

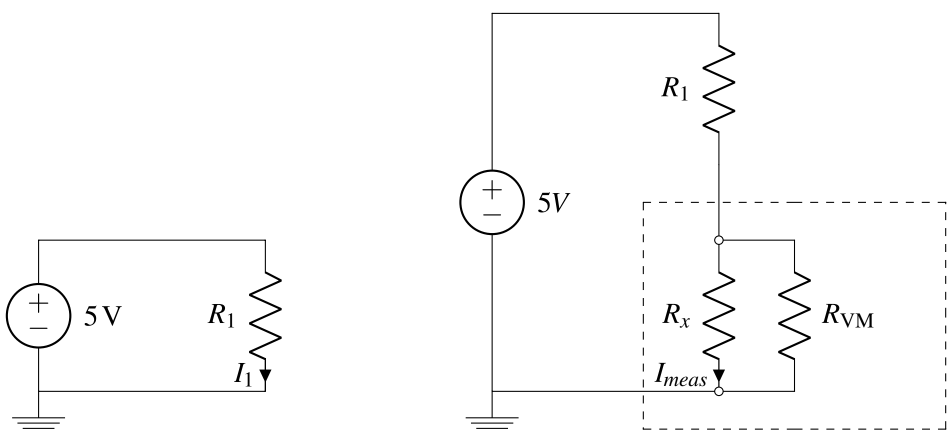


Figure 4: Circuits for Part (d) Left: Original circuit; Right: Circuit with the voltmeter connected as an ammeter.

Solution:
We start with the circuit on the left

$$I_1 = \frac{5\,\text{V}}{1\,\text{k}\Omega} = 5\,\text{mA}$$

For the circuit on the right, we start by computing $R_x||R_{\text{VM}}$.

$$R_x||R_{\text{VM}} = \frac{R_x R_{\text{VM}}}{R_x + R_{\text{VM}}} = \frac{1\,\Omega \cdot 1\,\text{M}\Omega}{1\,\Omega + 1\,\text{M}\Omega} \approx 1\,\Omega$$

Next, we compute the voltage across the $R_x||R_{\text{VM}}$ combination. Notice this circuit is again a voltage divider.

$$V_{R_{\text{VM}}} = \frac{R_x||R_{\text{VM}}}{R_1 + R_x||R_{\text{VM}}} \cdot 5\,\text{V} = \frac{1\,\Omega}{1\,\text{k}\Omega + 1\,\Omega} \cdot 5\,\text{V} = 0.004995\,\text{V}$$

The measured current is this voltage divided by the resistance R_x.

$$I_{\text{meas}} = \frac{V_{R_{\text{VM}}}}{R_x} = \frac{0.004\,995\,\text{V}}{1\,\Omega} \approx 5\,\text{mA}$$

(e) What is the minimum value of R_1 that ensures the difference between current measurement (I_{meas}) and the the actual value (I_1) stays within $\pm 10\%$ of I_1?

Solution:

Aain, we will only consider the case where the measured current is smaller than the actual current, because the series combination of R_1 and $R_x || R_{\text{VM}}$ can only create a resistor bigger than R_1. First let's symbolically represent what the outputs are in the two cases:

For the circuit without the ammeter connected:

$$I_1 = \frac{V_s}{R_1}$$

For the circuit with the ammeter connected:

$$R_{\text{VM}} || R_x = \frac{R_{\text{VM}} R_x}{R_{\text{VM}} + R_x}$$

$$V_{\text{VM}} = \frac{\frac{R_{\text{VM}} R_x}{R_{\text{VM}} + R_x}}{R_1 + \frac{R_{\text{VM}} R_x}{R_{\text{VM}} + R_x}} \cdot V_s$$

$$I_{\text{meas}} = \frac{V_{\text{VM}}}{R_x} = \frac{\frac{R_{\text{VM}}}{R_{\text{VM}} + R_x} \cdot V_s}{\frac{R_{\text{VM}} R_x}{R_{\text{VM}} + R_x} + R_1}$$

Now we need:

$$\frac{I_1 - I_{\text{meas}}}{I_1} \le \frac{1}{10}$$

$$\frac{\frac{V_s}{R_1} - \frac{\frac{R_{\text{VM}}}{R_{\text{VM}} + R_x} \cdot V_s}{\frac{R_{\text{VM}} R_x}{R_{\text{VM}} + R_x} + R_1}}{\frac{V_s}{R_1}} \le \frac{1}{10}$$

$$1 - \frac{\frac{R_{\text{VM}} R_1}{R_{\text{VM}} + R_x}}{\frac{R_{\text{VM}} R_x}{R_{\text{VM}} + R_x} + R_1} \le \frac{1}{10}$$

We will approximate $\frac{R_{\text{VM}} R_x}{R_{\text{VM}} + R_x} = R_x = 1\,\Omega$.

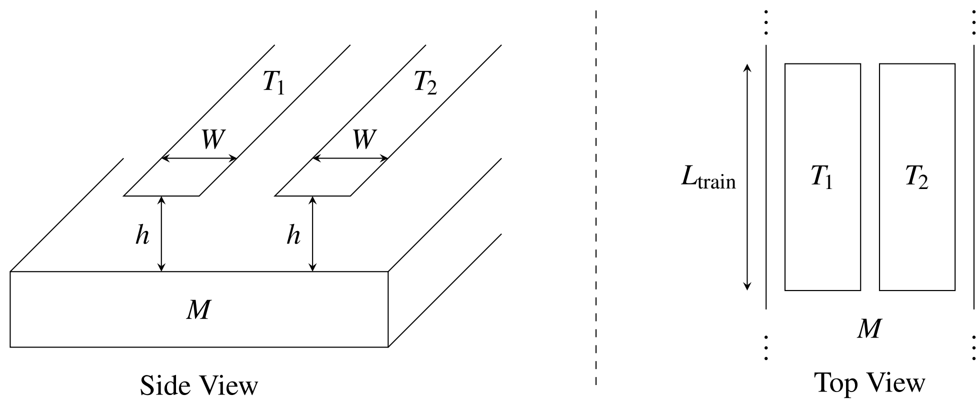
$$1 - \frac{R_1}{1 + R_1} = \frac{1}{1 + R_1} \le \frac{1}{10}$$

$$9\Omega \le R_1$$

4. Maglev Train Height Control System

One of the fastest forms of land transportation are trains that actually travel slightly elevated from the ground using magnetic levitation (or "maglev" for short). Ensuring that the train stays at a relatively constant height above its "tracks" (the tracks in this case are what provide the force to levitate the train and propel it forward) is critical to both the safety and fuel efficiency of the train. In this problem, we'll explore how maglev trains use capacitors to stay elevated. (Note that real maglev trains may use completely different and much more sophisticated techniques to perform this function, so if you get a contract to build such a train, you'll probably want to do more research on the subject.)

(a) As shown below, we put two parallel strips of metal (T_1, T_2) along the bottom of the train and we have one solid piece of metal (M) on the ground below the train (perhaps as part of the track).



Assuming that the entire train is at a uniform height above the track and ignoring any fringing fields (i.e., we can use the simple equations developed in lecture to model the capacitance), as a function of L_{train} (the length of the train), W (the width of T_1 and T_2), and h (the height of the train off of the track), what is the capacitance between T_1 and M? What is the capacitance between T_2 and M?

Solution:

The distance between the plates (T_1 & M or T_2 & M) is h. The area of the parallel plate capacitor is $A = WL_{\text{train}}$. Using the formula for capacitance of a parallel plate capacitor, we get:

$$C = \frac{\varepsilon A}{d}$$

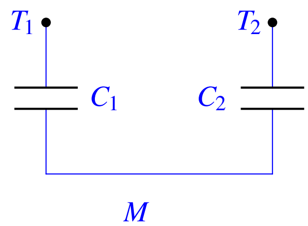
$$C_1 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between T}_1\text{ and M)}$$

$$C_2 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between T}_2\text{ and M)}$$

(b) Any circuit on the train can only make direct contact at T_1 and T_2. Thus, you can only measure the equivalent capacitance between T_1 and T_2. Draw a circuit model showing how the capacitors between T_1 and M and between T_2 and M are connected to each other.

Solution:

The capacitors C_1 and C_2 are in series. To realize this, let's consider the train circuit that is in contact with T_1 and T_2. If there is current entering plate T_1, the same current has to exit plate T_2. Thus, the circuit can be modeled as follows:



(c) Using the same parameters as in part (a), provide an expression for the equivalent capacitance between T_1 and T_2.

Solution:

Since the two capacitors are in series, the equivalent capacitance between T_1 and T_2 is given by:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

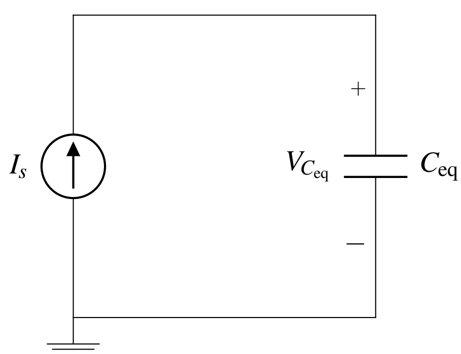
Thus, we get

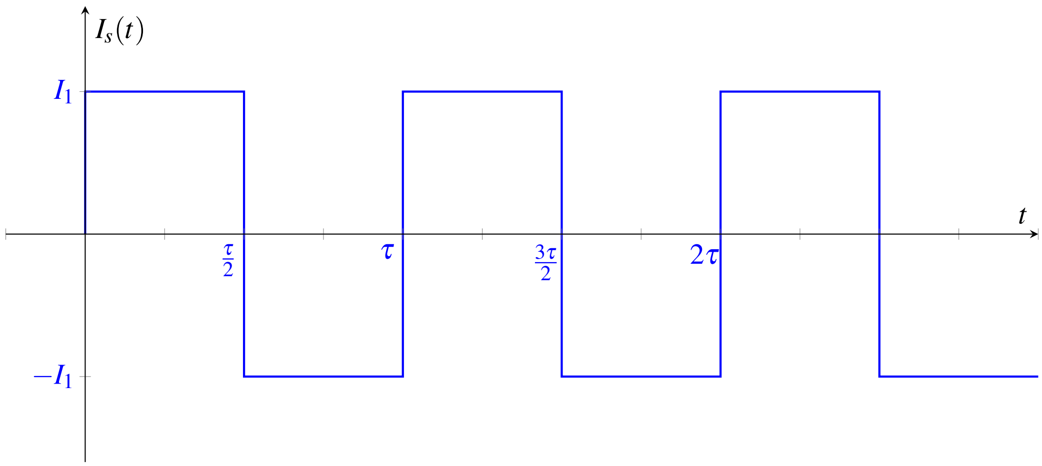
$$\frac{1}{C_{\text{eq}}} = \frac{h}{\varepsilon W L_{\text{train}}} + \frac{h}{\varepsilon W L_{\text{train}}}$$

$$C_{\text{eq}} = \frac{\varepsilon W L_{\text{train}}}{2h}$$

(d) We want to build a circuit that creates a voltage waveform with an amplitude that changes based on the height of the train. Your colleague recommends you start with the circuit as shown below, where I_s is a periodic current source, and C_{eq} is the equivalent capacitance between T_1 and T_2. The graph below shows I_s, a square wave with period τ and amplitude I_1, as a function of time.

Find an equation for and draw the voltage $V_{C_{\text{eq}}}(t)$ as a function of time. Assume the capacitor C_{eq} is discharged at time $t = 0$, so $V_{C_{\text{eq}}}(0) = 0\,\text{V}$.





Solution: We know the rate of change of voltage across a capacitor is related to the the current into the capacitor. That is:

$$I_{C_{\text{eq}}} = C_{\text{eq}} \frac{dV_{C_{\text{eq}}}}{dt}$$

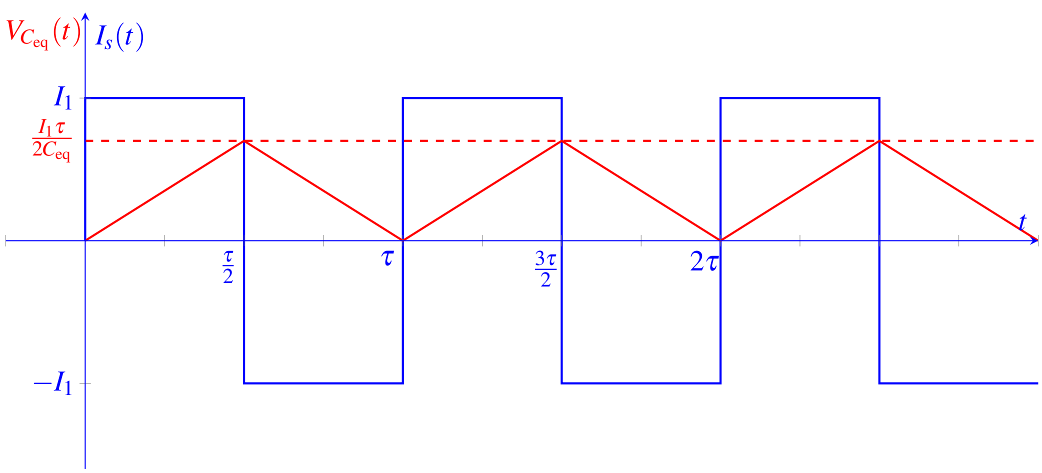
From KCL, we know $I_{C_{\text{eq}}} = I_s$. Then:

$$I_{C_{\text{eq}}} = I_s = C_{\text{eq}} \frac{dV_{C_{\text{eq}}}}{dt} \implies \frac{dV_{C_{\text{eq}}}}{dt} = \frac{I_s}{C_{\text{eq}}}$$

Since I_s is periodic, we can apply the procedure detailed in Note 17, Section 17.2.1 to get the following equation for $V_{C_{\text{eq}}}(t)$ for the first period, which repeats for subsequent periods. We recall that the capacitor is uncharged at $t = 0$ so that $V_{C_{\text{eq}}}(0) = 0\,\text{V}$.

$$V_{C_{\text{eq}}}(t) = \begin{cases} \frac{I_1}{C_{\text{eq}}} t & \text{when } 0 \le t \le \frac{\tau}{2} \\ \frac{-I_1}{C_{\text{eq}}} \left(t - \frac{\tau}{2}\right) + \frac{I_1 \tau}{2C_{\text{eq}}} & \text{when } \frac{\tau}{2} < t \le \tau \end{cases}$$

Given this equation for the output voltage, $V_{C_{\text{eq}}}(t)$, as a function of the current, I_s, we can draw what the output waveform should look like.



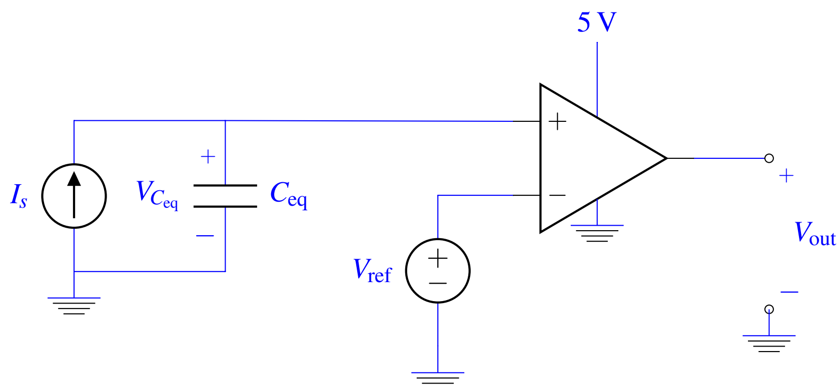
(e) We now want to develop an indicator that alerts us when the train is too high above the tracks. We want to output a series of 5 V pulses that can be used to drive a horn when the train is above 1 cm, and not output anything when the train is below 1 cm.

We will assume the train has length $L_{\text{train}} = 100\,\text{m}$ and that the metals, T_1 and T_2, have width $W = 1\,\text{cm}$ and permitivity $\varepsilon = 8.85 \times 10^{-12}\,\text{F}\,\text{m}^{-1}$.

Design a circuit using a square wave current source (i.e. I_s in part (d)) with period $\tau = 1\,\mu\text{s}$ and pulses of amplitude $I_1 = 1\,\text{mA}$, a comparator, and any number of voltage sources to implement this function. **Hint: you should use the circuit you analyzed in part (d).**

Solution:

The circuit is shown below:



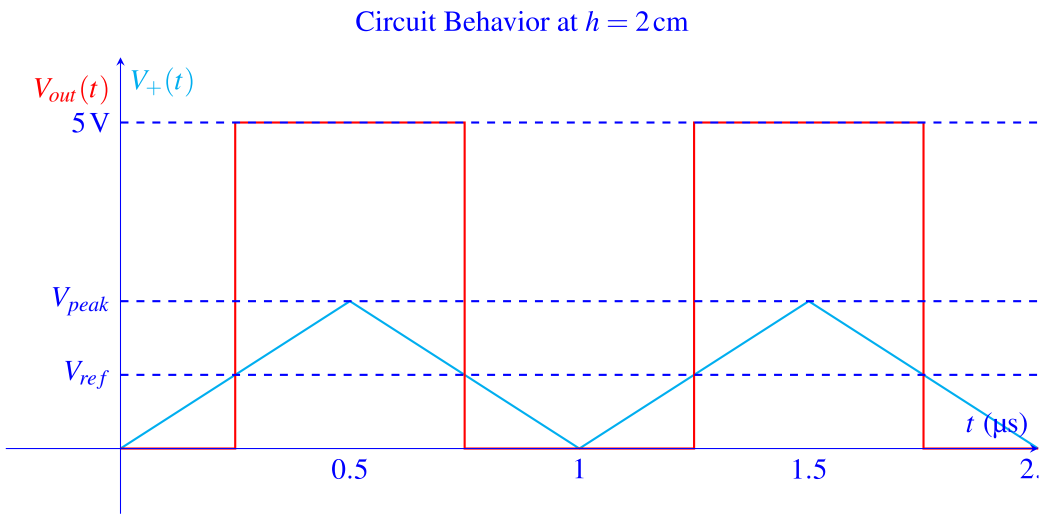
From the choice of supply voltages, we see that $V_{out} = 5\,\text{V}$ when $V_+ > V_-$.

We know the amplitude of $V_{C_{\text{eq}}}(t)$ when $h = 1\,\text{cm}$ is:

$$\frac{I_1 \tau}{2C_{\text{eq}}} = \frac{1\,\text{mA} \cdot 1\,\mu\text{s}}{2 \cdot C_{\text{eq}}} = \frac{1\,\text{mA} \cdot 1\,\mu\text{s}}{2 \cdot \frac{\varepsilon W L_{\text{train}}}{2h}} = \frac{1\,\text{mA} \cdot 1\,\mu\text{s} \cdot h}{\varepsilon W L_{\text{train}}} = \frac{1\,\text{mA} \cdot 1\,\mu\text{s} \cdot 1\,\text{cm}}{8.85 \times 10^{-12}\,\text{F}\,\text{m}^{-1} \cdot 1\,\text{cm} \cdot 100\,\text{m}} = 1.13\,\text{V}$$

Thus we can set V_{ref} to this peak value assuming the train is 1 cm above the ground. If the train's height is larger than 1 cm, the peak voltage rises, and we continue to get pulses. If the train's height is below 1 cm, the peak value is less than 1.13 V preventing any pulses from the output of the circuit.

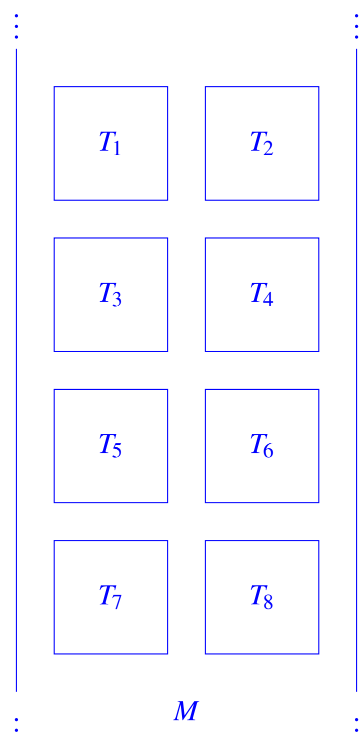
As an example, let's suppose the train's height is 2 cm. Then we would observe the following output for V_{out}. Note that the x-axis is in µs, that $V_{ref} = 1.13\,\text{V}$ as we found before, and that $V_{peak} = 2 \cdot V_{ref} = 2.26\,\text{V}$. The cyan waveform is what we measure at V_+, and the red waveform is the 5 V pulse generated at V_{out}.



(f) So far we've assumed that the height of the train off of the track is uniform along its entire length, but in practice, this may not be the case. Suggest and sketch a modification to the basic sensor design (i.e., the two strips of metal T_1 and T_2 along the entire bottom of the train) that would allow you to measure the height at the train at 4 different locations.

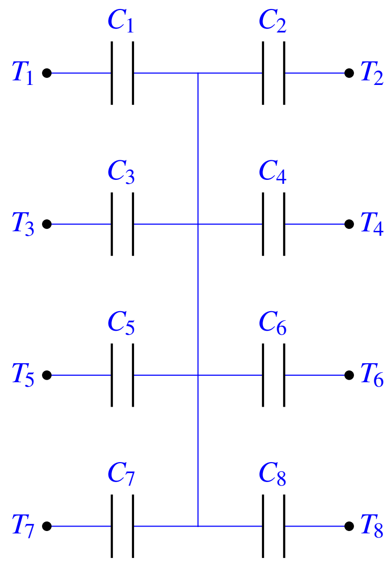
Solution:

One possible solution is shown below. Here we divide the T_1 and T_2 strips into many shorter strips.

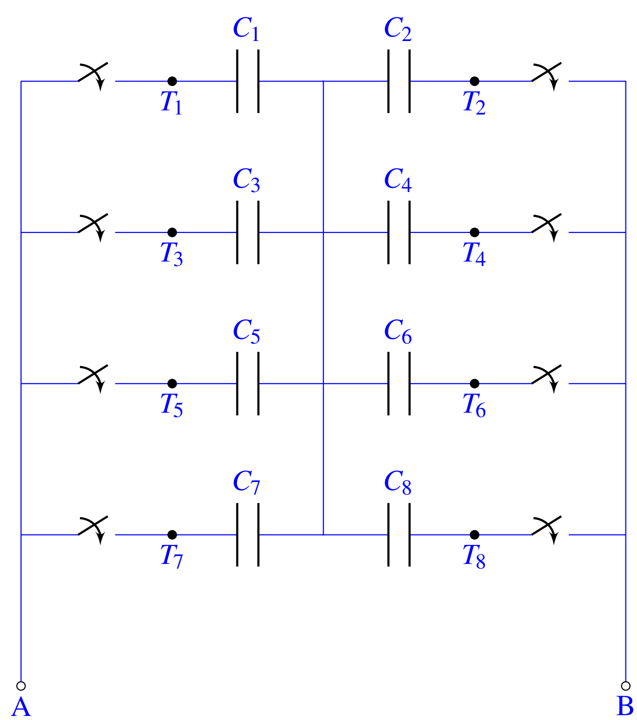


Top View

One important thing to note about this circuit is that it works only if extra care is taken during the capacitance measurement circuit. The equivalent model for this is:

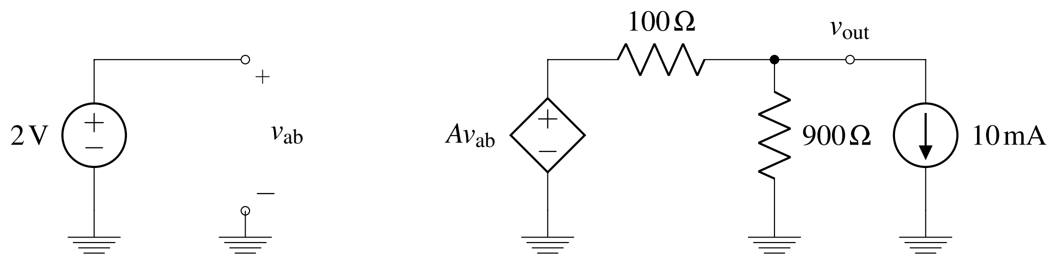


Because all of the caps are connected together by the rail under the train, we need to have a way to select only some of the caps. We can accomplish this by having seperate switches on each T, so that you can measure the capacitance between only two terminals (like T_1 & T_2) and so that the effect of other capacitors is nullified. This is shown below, where the points A and B are where you connect your previous circuit.



5. Superposition with a Dependent Source

Given $A = 5$, find the voltage v_{out} indicated in the circuit diagram below using superposition.



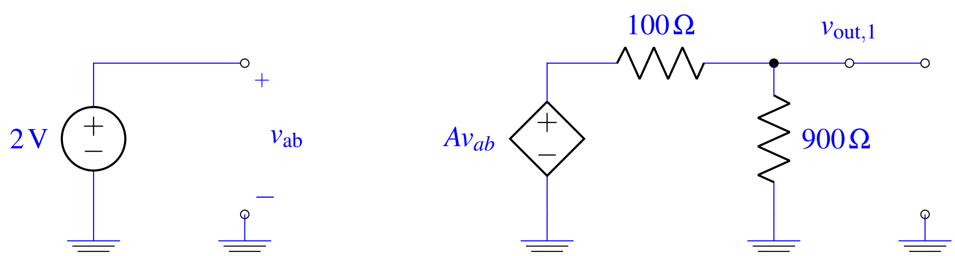
Solution:

$$v_{out} = 8.1\,\text{V} \tag{1}$$

First, we note that the voltage $v_{ab} = 2\text{V}$ since it is measuring across the voltage source. Our voltage-controlled source will then be $Av_{ab} = 5(2\text{V}) = 10\text{V}$.

Now, consider the circuits obtained by:

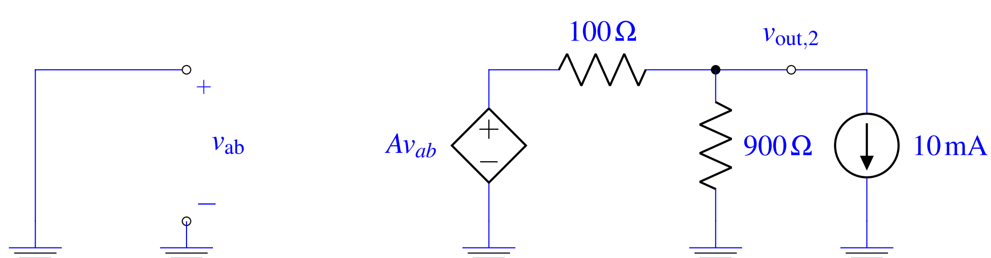
(a) Turning off the current source:



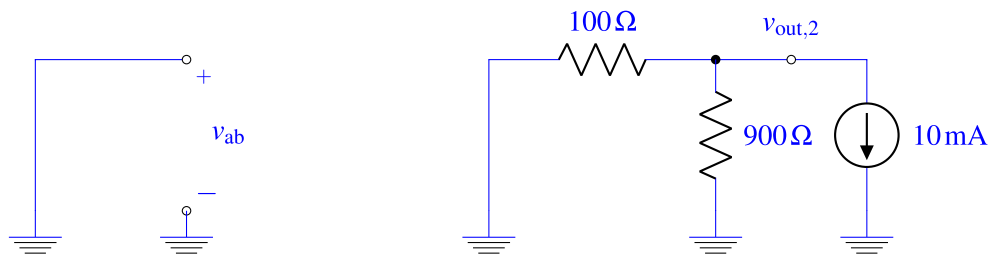
In the above circuit, no current is going to flow through the rightmost branch, as it is an open circuit. Thus this is just a 10 V voltage source connected to a 100 Ω resistor and a 900 Ω resistor in series, so we use the voltage divider formula.

$$v_{out,1} = \frac{900}{100+900} 10\,\text{V} = 9\,\text{V}$$

(b) Turning off the voltage source:



Since our independent voltage source is set to zero volts, the voltage V_{ab} will be zero and thus the voltage-controlled source will also be zero. We can redraw the circuit again to reflect this.



Now, looking at the circuit on the right, we have the two resistors in parallel and connected to the current source. The equivalent resistance is given below.

$$R_{eq} = \frac{900(100)}{900+100} = 90\,\Omega \tag{2}$$

Using this equivalent resistance, we find the voltage at $v_{out,2}$ using Ohm's law.

$$v_{out,2} = (-10\,\text{mA})(90\,\Omega) = -0.9\,\text{V} \tag{3}$$

Note that this is a negative voltage since the current is flowing up through the resistors, resulting in a potential at $v_{out,2}$ that is lower than that of the ground node.

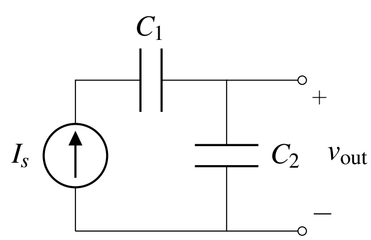
Now, applying the principle of superposition, we can solve for v_{out}.

$$v_{out} = v_{out,1} + v_{out,2} = 9\,\text{V} - 0.9\,\text{V} = 8.1\,\text{V} \tag{4}$$

-

6. Current Sources And Capacitors

For the circuit given below, give an expression for $v_{out}(t)$ in terms of I_s, C_1, C_2, and t. Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is 0 V.



Solution:

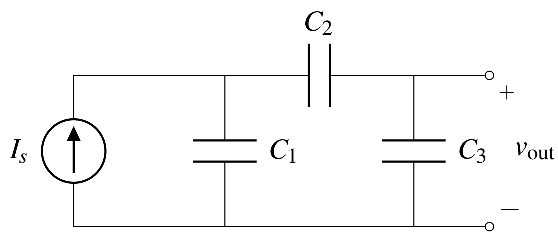
By KCL, the current I_s flowing through C_1 must be the current flowing through C_2. $v_{out}(0) = 0$ because all capacitors are initially uncharged.

$$I_s = C_2 \frac{dv_{out}(t)}{dt}$$

$$v_{out}(t) = \int \frac{I_s}{C_2} dt = \frac{I_s t}{C_2} + v_{out}(0) = \frac{I_s t}{C_2}$$

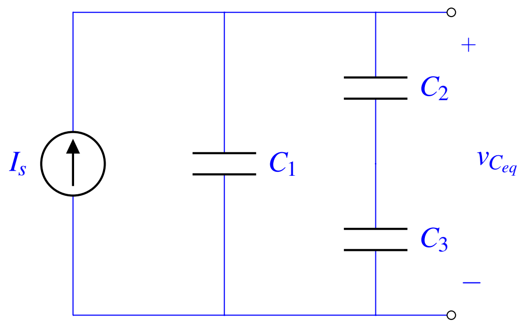
7. More Current Sources And Capacitors

For the circuit given below, give an expression for $v_{out}(t)$ in terms of I_s, C_1, C_2, C_3, and t. Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is 0 V.

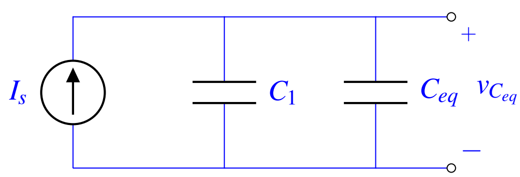


Solution:

Instead of finding v_{out} directly, let's first find the voltage $v_{C_{eq}}$ across C_2 and C_3.



To do this, we replace C_2 and C_3 with their equivalent capacitance $C_{eq} = C_2 \parallel C_3 = \frac{C_2 C_3}{C_2 + C_3}$.



We know that to solve for $v_{C_{eq}}$, we can find the equivalent capacitance of C_1 and C_{eq} first, which is $C_1 + C_{eq}$. Since the capacitors are initially uncharged, $v_{C_{eq}}(0) = 0$.

$$v_{C_{eq}}(t) = \int \frac{I_s}{C_1 + C_{eq}} dt = \frac{I_s t}{C_1 + C_{eq}} + v_{C_{eq}}(0) = \frac{I_s t}{C_1 + C_{eq}}$$

Now that we know that voltage across the equivalent capacitor C_{eq}, we can find the current flowing through the equivalent capacitor C_{eq}.

$$i_{C_{eq}}(t) = C_{eq} \frac{dv_{C_{eq}}(t)}{dt} = \frac{C_{eq} I_s}{C_1 + C_{eq}}$$

Note that the current $i_{C_{eq}}$ is equal to the current flowing through C_3 since C_2 and C_3 were originally connected in series.

$$i_{C_3}(t) = i_{C_{eq}}(t) = \frac{C_{eq} I_s}{C_1 + C_{eq}}$$

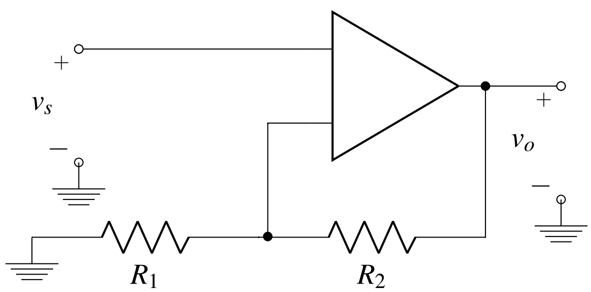
Since v_{out} is the voltage across the capacitor C_3, we integrate to find v_{out}. Again, since all capacitors are initially uncharged, $v_{\text{out}}(0) = 0$.

$$i_{C_3}(t) = C_3 \frac{dv_{\text{out}}(t)}{dt}$$

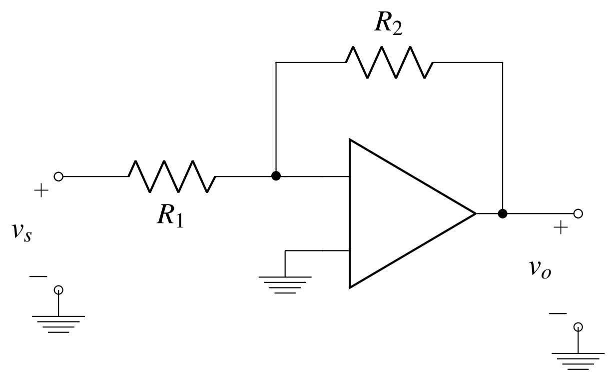
$$v_{\text{out}}(t) = \int \frac{C_{eq} I_s}{C_3(C_1 + C_{eq})} dt = \frac{C_{eq} I_s t}{C_3(C_1 + C_{eq})} + v_{\text{out}}(0) = \frac{\frac{C_2 C_3}{C_2 + C_3} I_s t}{C_3\left(C_1 + \frac{C_2 C_3}{C_2 + C_3}\right)} = \frac{C_2 I_s t}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

8. Basic Amplifier Building Blocks

The following amplifier stages are used often in many circuits and are well known as (a) the non-inverting amplifier and (b) the inverting amplifier.



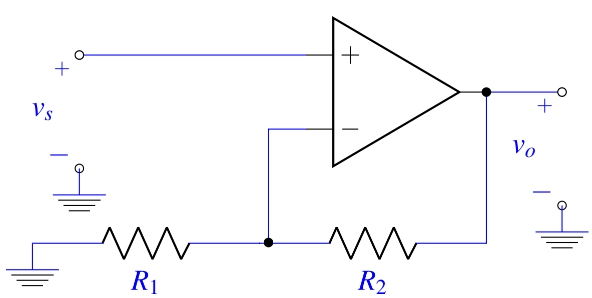
(a)



(b)

(a) Label the input terminals of the op-amp labeled (a), so that it is in negative feedback. Then derive the voltage gain ($A_v = \frac{v_o}{v_s}$) of the non-inverting amplifier using the Golden Rules. Explain the origin of the name of the amplifier.

Solution:



The $+,-$ should be labeled on the top and bottom of the op amp, respectively.

There are many ways to solve these circuits; here are some:

Method 1: The voltage at the positive input terminal is v_s, so by the Golden Rules, the op-amp will act such that the voltage at the negative input terminal also becomes v_s. Therefore, the voltage drop across R_1 is v_s, so there is a current of $i = \frac{v_s}{R_1}$ through resistor R_1. Since no current flows into the negative input terminal (by the Golden Rules), this current of i must flow through R_2 (by KCL at the inverting input). Thus, the voltage drop across R_2 is $V_2 = i \cdot R_2 = v_s \left(\frac{R_2}{R_1}\right)$. Therefore, v_o is v_s plus the voltage drop across R_2:

$$v_o = v_s + v_s \left(\frac{R_2}{R_1}\right) = v_s \left(\frac{R_1 + R_2}{R_1}\right)$$

Method 2: Since there is no current flowing into the negative input terminal (by the Golden Rules), notice that the resistors R_1 and R_2 form a voltage divider between the output v_o and ground. The negative input terminal sees the output of this voltage divider:

$$u_- = v_o \left(\frac{R_1}{R_1 + R_2} \right)$$

But $u_- = u_+ = v_s$ by the Golden Rules, so we have:

$$v_o \left(\frac{R_1}{R_1 + R_2} \right) = v_s \implies v_o = v_s \left(\frac{R_1 + R_2}{R_1} \right)$$

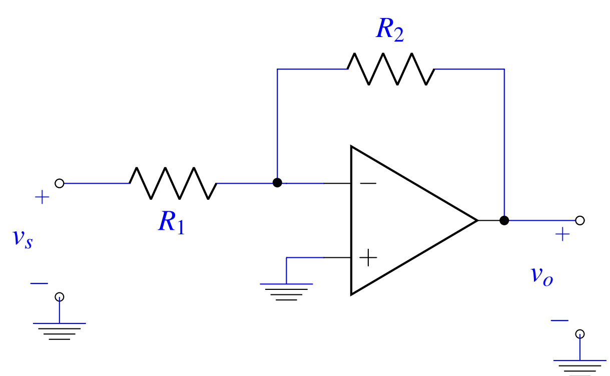
Therefore, the gain of this amplifier is:

$$A_v = \frac{v_o}{v_s} = \frac{R_1 + R_2}{R_1}$$

This is called an *non-inverting amplifier* because the gain A_v is positive – it does not invert the input signal (in contrast to the amplifier in the next part of this problem).

(b) Label the input terminals of the op-amp labeled (b), so that it is negative feedback. Then derive the voltage gain ($A_v = \frac{v_o}{v_s}$) of the inverting amplifier using the Golden Rules. Explain the origin of the name of the amplifier.

Solution:



The $+, -$ should be labelled on the bottom and top of the op amp, respectively.

Here is one way to solve for the gain:

Since the potential at the positive input terminal is $u_+ = 0$, the op-amp will act such that the potential at the negative input terminal is $u_- = 0$ as well (by the Golden Rules). Now, by KCL at the node with potential u_-:

$$\frac{v_s - 0}{R_1} + \frac{v_o - 0}{R_2} = 0$$

Solving this yields:

$$v_o = -\left(\frac{R_2}{R_1} \right) v_s$$

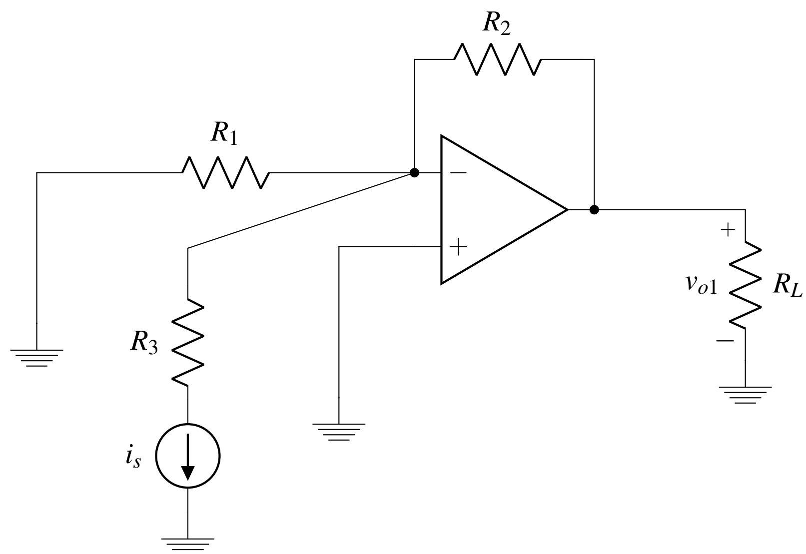
Thus, the voltage gain of this amplifier circuit is:

$$A_v = \frac{v_o}{v_s} = -\frac{R_2}{R_1}$$

This is called an *inverting amplifier* because the voltage gain A_v is *negative*, meaning it "inverts" its input signal.

9. Amplifier with Multiple Inputs

(a) Use the Golden Rules to find v_{o1} for the circuit below.



Solution:

Applying the Golden Rules, we know that the positive and negative terminals must be at the same voltage. Thus, the voltage at the negative terminal of the op-amp is 0. The voltage drop across R_1 is 0 and no current flows through it. In addition, no current flows into the op-amp from the negative terminal due to its infinite input resistance (the negative terminal is connected to an "open" circuit).

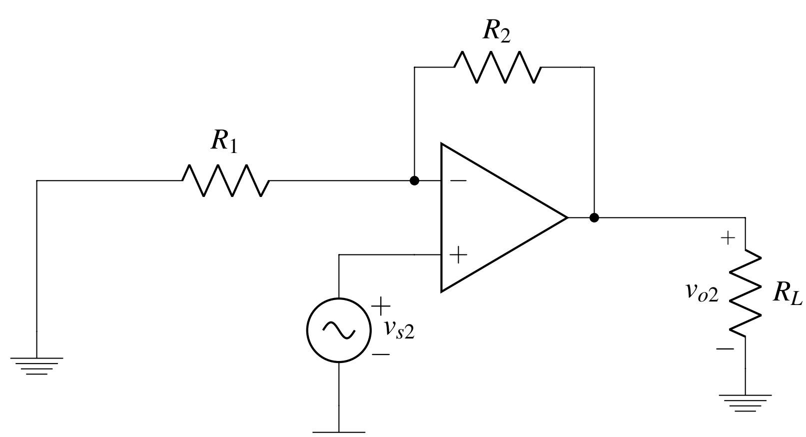
By KCL at the negative terminal of the op-amp, this means that the current going through R_3 and R_2 is i_s. Taking the positive terminal of R_2 to be on the right, the voltage drop across R_2 is v_{o1}. By Ohm's law, we conclude:

$$\frac{v_{o1}}{R_2} = i_s$$

Rearranging we get:

$$v_{o1} = i_s \cdot R_2$$

(b) Use the Golden Rules to find v_{o2} for the circuit below.



Solution:
Applying the Golden Rules, we know that the positive and negative terminals must be at the same voltage. Thus, the voltage at the negative terminal of the op-amp is $V^- = v_{s2}$. In addition, since no current can enter into the negative terminal of the op-amp, R_1 and R_2 are in series. This means that the voltage at the negative terminal of the op-amp can be expressed in terms of v_{o2} using the voltage divider formula:

$$v^- = v_{o2}\left(\frac{R_1}{R_1+R_2}\right)$$

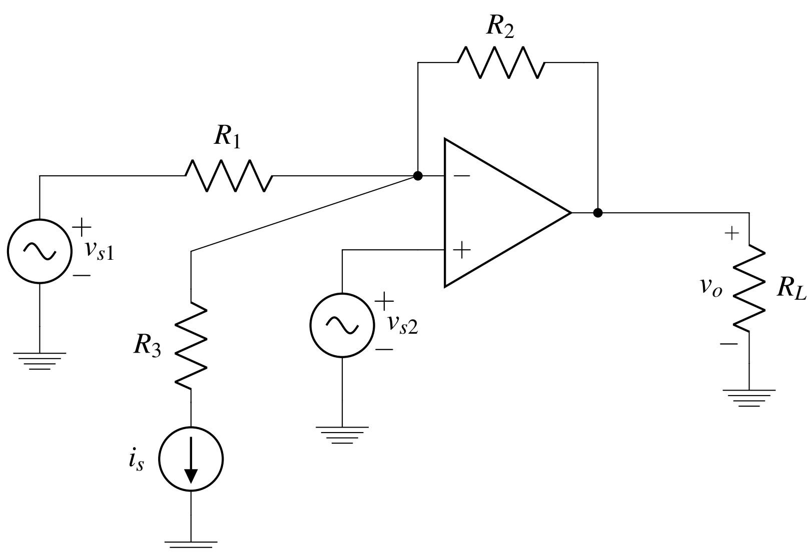
We also know that $v^- = v_{s2}$ and conclude:

$$v_{s2} = v_{o2}\left(\frac{R_1}{R_1+R_2}\right)$$

After rearranging, we have:

$$v_{o2} = v_{s2}\left(\frac{R_2}{R_1}+1\right)$$

(c) Use the Golden Rules to find the output voltage v_o for the circuit shown below.



Solution:
Applying the Golden Rules we know that the positive and negative terminals must be at the same voltage. Thus, the voltage at the negative terminal of the op-amp is $v^- = v_{s2}$. Then we write a KCL equation at the node connected to the minus terminal of the op-amp (recalling that no current flows into or out of the op-amp's terminals). All currents are defined as flowing out of the node:

$$i_{R_1} + i_{R_2} + i_{R_3} = 0$$

Because of the independent current source, we know:

$$i_{R_3} = i_s$$

By Ohm's law, we know:

$$i_{R_1} = \frac{v^- - v_{s1}}{R_1}$$

and

$$i_{R_2} = \frac{v^- - v_o}{R_2}$$

Then, substituting back into the original KCL equation, we have:

$$\frac{v^- - v_{s1}}{R_1} + \frac{v^- - v_o}{R_2} + i_s = 0$$

and substituting $v^- = v_{s2}$, we have:

$$\frac{v_{s2} - v_{s1}}{R_1} + \frac{v_{s2} - v_o}{R_2} + i_s = 0$$

which we rearrange to find v_o, giving:

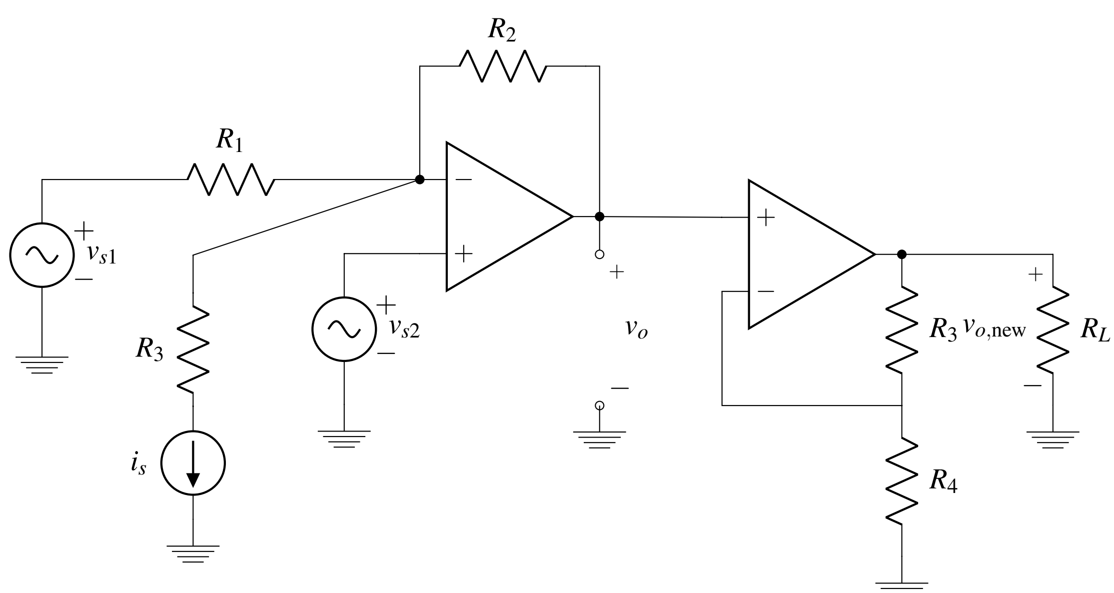
$$v_o = v_{s2}\left(1 + \frac{R_2}{R_1}\right) + i_s \cdot R_2 - \left(\frac{R_2}{R_1}\right) v_{s1}$$

(d) Use superposition and the answers to the first few parts of this problem to check your work.

Solution: Using superposition we can analyze the circuit leaving only one source on at a time. If we leave on v_{s1} and turn off v_{s2} and i_{s3}, then we have the first circuit. If we leave on i_{s3} and turn off v_{s1} and v_{s2}, then we have the second circuit. If we leave on v_{s2} and turn off v_{s1} and i_{s3}, then we have the third circuit. From this we can see that v_o is the sum from the solutions in part a, c, and e.

$$v_o = -\frac{R_2}{R_1} v_{s1} + i_{s3} R_2 + v_{s2} \frac{R_2 + R_1}{R_1}$$

(e) Now add a second stage as shown below. What is $v_{o,\text{new}}$? Does v_o change between part (c) and this part? Does the voltage $v_{o,\text{new}}$ depend on R_L?



Solution:

Adding the second stage does not change the voltages in the first stage. This is because the circuit connected to the positive and negative terminals of the first stage op-amp "sees" an open circuit/infinite input resistance in the op-amp.

Hence v_o remains unchanged from part (c).

$$v_o = -\left(\frac{R_2}{R_1}\right) v_{s1} + i_s \cdot R_2 + v_{s2}\left(\frac{R_2+R_1}{R_1}\right)$$

By the Golden Rules, the negative terminal of the second op-amp must have the same voltage as the plus terminal, which is v_{o1}. No current can flow into the negative terminal, so R_3 and R_4 are in series and have the same current, so we know:

$$\frac{v_o}{R_4} = \frac{v_{o,new} - v_o}{R_3}$$

Therefore:

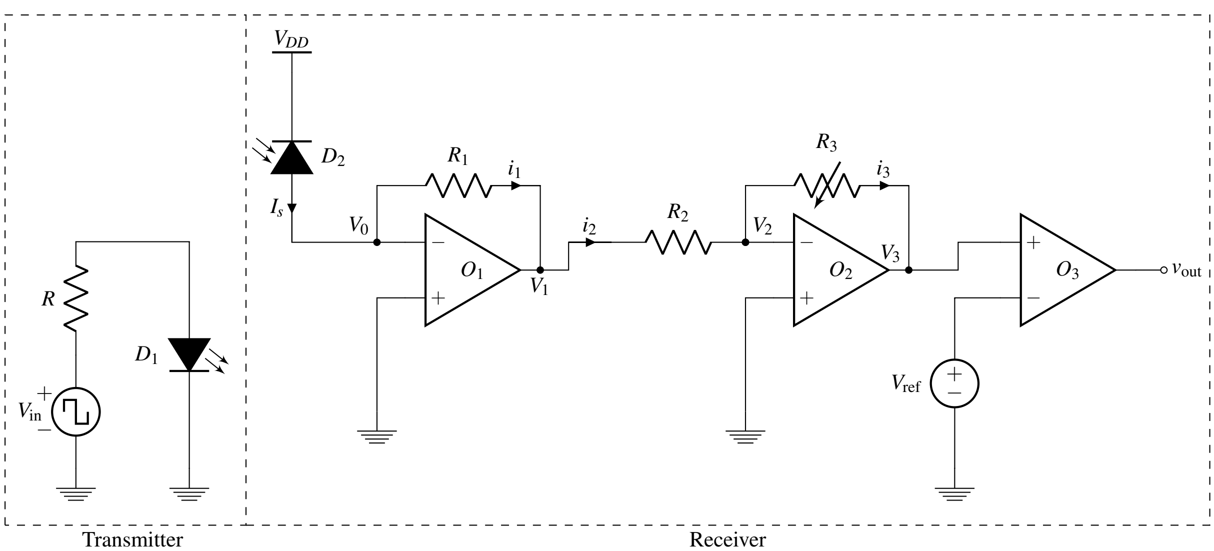
$$v_{o,new} = \left(\frac{R_3+R_4}{R_4}\right) v_{o1} = \frac{R_3+R_4}{R_4}\left(-\frac{R_2}{R_1} \cdot v_{s1} + i_s \cdot R_2 + v_{s2} \cdot \frac{R_2+R_1}{R_1}\right)$$

Note that you could have directly used the non-inverting amplifier gain formula $(1+\frac{R_3}{R_4})$for this extra stage.

The output voltage **does not** depend on the load resistance R_L, since it is set by the dependent voltage source inside the op-amp. Remember that a voltage source will provide any amount of current necessary while maintaining its voltage constant. **That is the beauty of op-amps:** they provide isolation between stages because of the open circuit at the input and they get rid of the loading effect, since they can maintain the output voltage constant regardless of the load value.

10. Wireless Communication With An LED

In this question, we are going to analyze the system shown in the figure below. It shows a circuit that can be used as a wireless communication system using visible light (or infrared, very similar to remote controls).



The element D_1 in the transmitter is a light-emitting diode (LED). An LED is an element that emits light and whose brightness is controlled by the current flowing through it. You can recall controlling the light emitted by an LED using your MSP430 in Touchscreen Lab 1. In our circuit, the current across the LED, hence its brightness, can be controlled by choosing the applied voltage V_{in} and the value of the resistor R. In the receiver, the element labeled as D_2 is a reverse biased solar cell. You can recall using an ambient light sensor in Imaging Labs 1 to 3 as a light-controlled current source. We will denote the current supplied by the solar cell by I_S. In this circuit, the LED D_1 is used as a means for transmitting information with light, and the reverse-biased solar cell D_2 is used as a light receiver to see if anything was transmitted.

Remark: In Imaging Lab 3, we talked about how non-idealities, such as background light, affect the performance of a system that does light measurements. In this question, we will assume ideal conditions, that is, there is no source of light around except for the LED.

In our system, we define two states for the transmitter: the *transmitter is sending something* when they turn on the LED and the *transmitter is not sending anything* when they turn off the LED. On the receiver side, the goal is to convert the current I_S generated by the solar cell into a voltage and amplify it, so that we can read the output voltage V_{out} to see if the transmitter was sending something or not. The circuit implements this operation through a series of op-amps. It might look look complicated at first glance, but we can analyze it one section at a time.

(a) Currents i_1, i_2 and i_3 are labeled on the diagram. Assuming the Golden Rules hold, is $I_S = i_1$? $i_1 = i_2$? $i_2 = i_3$? Treat the solar cell as an ideal current source.

Solution:

We use the Golden Rules, which say that in an op-amp, no current flows into or out of V_+ or V_-. Therefore we can use KCL at node V_0 and V_2 to conclude that $\boxed{I_S = i_1}$ and $\boxed{i_2 = i_3}$. However, if $I_S \neq 0$ and $R_1 \neq R_2$, then $\boxed{i_1 \neq i_2}$. This is because $V_0 = V_2 = 0\,\text{V}$ and V_1 is some non-zero voltage. If $R_1 \neq R_2$, then the currents flowing through them are different.

(b) Use the Golden Rules to find V_0, V_1, V_2 and V_3 in terms of I_S, R_1, R_2 and R_3.

Hint: Solve for them from left to right, and remember to use the Golden Rules.

Solution:

Using the Golden Rules, we know that $\boxed{V_0 = 0\,\text{V}}$. Using Ohm's law, we know that $V_0 - V_1 = i_1 R_1$. From the previous part, we know that $i_1 = I_S$. Thus, we get $\boxed{V_1 = -I_S R_1}$. Using the Golden Rules again, we get $\boxed{V_2 = 0\,\text{V}}$. Using Ohm's law and the KCL result from the previous part, we get the following equations:

$$\begin{aligned} V_1 - V_2 &= i_2 R_2 \\ V_2 - V_3 &= i_3 R_3 \\ i_2 &= i_3 \end{aligned}$$

Solving them, we get $\boxed{V_3 = I_S \dfrac{R_1 R_3}{R_2}}$.

(c) In the previous part, how could you check your work to gain confidence that you got the right answer?

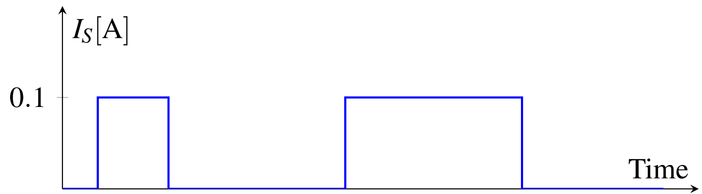
Solution:

One sanity check is checking that your answer has the right units ($\text{V} = \text{A} \cdot \frac{\Omega \cdot \Omega}{\Omega} = \text{A} \cdot \Omega$).

Also notice that R_2 and R_3 form a voltage divider since no current flows into the negative terminal of O_2. Thus, we can check that the voltage divider equation holds:

$$V_2 - V_1 = (V_3 - V_1) \frac{R_2}{R_2 + R_3}$$

(d) Now, assume that the transmitter has chosen the values of V_{in} and R to control the intensity of light emitted by LED, such that when the *transmitter is sending something*, I_{S} is equal to $0.1\,\text{A}$ and when the *transmitter is not sending anything*, I_{S} is equal to $0\,\text{A}$. The following figure shows a visual example of how this current I_{S} might look like as time changes (note that this is just to help you visualize the shape of the current supplied by the solar cell).



For the receiver, suppose $V_{\text{ref}} = 2\,\text{V}$, $R_1 = 10\,\Omega$, $R_2 = 1000\,\Omega$, and the supply voltages of the op-amps are $V_{\text{DD}} = 5\,\text{V}$ and $V_{\text{SS}} = -5\,\text{V}$. Pick a value of R_3 such that V_{out} is V_{DD} when the *transmitter is sending something* and V_{SS} when the *transmitter is not sending anything*?

Solution:

We want $v_{\text{out}} = V_{DD}$, when $V_3 - V_{\text{ref}} > 0\,\text{V}$ when $I_S = 0.1\,\text{A}$ and $v_{\text{out}} = V_{SS}$, when $V_3 - V_{\text{ref}} < 0\,\text{V}$ when $I_S = 0\,\text{A}$. We plug the known resistor values into the equation in the previous part to get $V_3 = I_S \frac{R_3}{100}$. When $I_S = 0\,\text{A}$, $v_{\text{out}} = -2\,\text{V} < 0\,\text{V}$. When $I_S = 0.1\,\text{A}$, $v_{\text{out}} = \frac{R_3}{1000} - V_{\text{ref}} > 0\,\text{V}$. Thus, $\boxed{R_3 > 2000\,\Omega}$.

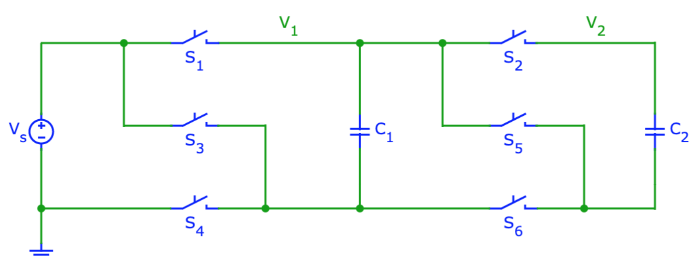
(e) In the previous part, how could you check your work to gain confidence that you got the right answer?

Solution:

We can check if the answer makes sense. We know that O_2 serves as an inverting amplifier and that V_1 is negative (since there is a voltage drop from $V_0 = 0\,\text{V}$). Thus, we want it to amplify V_1 until it is higher than V_{ref}, so R_3 should be bigger than R_2.

11. Voltage Booster

We have made extensive use of resistive voltage dividers to reduce voltage. What about a circuit that boosts voltage to a value greater than the supply $V_S = 5V$? We can do this with capacitors!



(a) In the circuit above switches S1, S2, S4 and S6 are initially closed and switches S3 and S5 open. Calculate voltages V_1 and V_2.

Solution:

In this setting, the two capacitors in parallel like so:

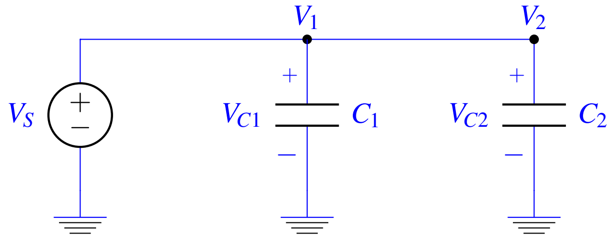


Figure 5: Phase 1

Hence,

$$V_1 = V_2 = V_S = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}.$$

(b) Now, after the capacitors are charged, switches S1, S2, S4 and S6 are opened and switches S3 and S5 closed. Calculate the new voltages V_1 and V_2.

Solution: In phase 2 notice that capacitors C_1 and C_2 have been switched, in a way that the "+" plates are floating i.e. there is no discharge path from nodes V_1 and V_2 to ground. The corresponding schematic is now the following:

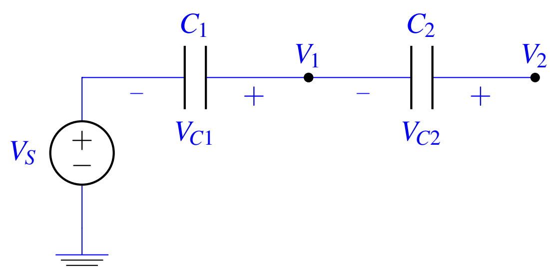


Figure 6: Phase 2

This means that the charge on these plates is going to be preserved so we will have for C_1:

$$Q_1 = V_S C_1 = (V_1 - V_S)C_1 \tag{5}$$
$$V_1 = 2V_S = 10V!! \tag{6}$$

Similarly, for C_2:

$$Q_2 = V_S C_2 = (V_2 - V_1)C_2 \tag{7}$$
$$V_1 = V_S + V_1 = 3V_S = 15V!! \tag{8}$$

12. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...
Then I went to homework party for a few hours, where I finished the homework.