Review: Eigenvalues and Eigenvectors

- Let's start with determinants!
  - Every square matrix has a determinant.
  - Has a physical interpretation as the volume of a parallelipiped enclosed by the columns of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ G & H & i \end{bmatrix} \Rightarrow \det(A) = a\left(\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right)\right) - b\left(\det\left(\begin{bmatrix} d & f \\ G & i \end{bmatrix}\right)\right) + c\left(\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)\right)$$

$$= a(ei - fh) - b(di - fG) + c(dh - eg)$$

$$= aei - afh - bdi + bfG + cdh - ceg$$

Same!
(Two ways to look at it)

<u>Or</u> try this for 3x3 matrices:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ G & H & i \end{bmatrix} \quad aei + bfG + cdH$$

$$-$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ G & H & i \end{bmatrix} \quad (ceG + bdi + afh)$$

$$\left.\begin{matrix} aei + bfG + cdH \\ - \\ (ceG + bdi + afh) \end{matrix}\right\} = aei + bfG + cdH - ceG - bdi - afh$$

We really care when $\det(A) = 0$

why? because $\boxed{\det(A) = 0 \iff A \text{ is } \underline{\text{not}} \text{ invertible}}$

$(\Rightarrow)$

$\text{Det}(A) = 0 \Rightarrow$ volume of parallelipiped is zero

$\Rightarrow$ one or more columns are colinear / dependent

$\Rightarrow$ A is not invertible

$(\Leftarrow)$

<u>Fact</u>: $A \xrightarrow[\text{elimination}]{\text{Gaussian}} U$ ← rref $\qquad \text{Det}(A) = \pm \text{Det}(U)$

A that is not invertible $\Rightarrow$ U will have a row of all zeros.

$\Rightarrow \det(A) = 0$

<u>Fact</u>: $\text{Det}(A) = \prod_i \lambda_i$

↖ product of eigenvalues

Eigenvalues & Eigenvectors

$\det(A - \lambda I) = 0$ ← where did this come from?

Definition of e-vecs and e-vals:

A square matrix has eigenvector $\vec{v}$ with eigenvalue λ if

$$A\vec{v} = \lambda\vec{v} \quad , \text{ where } \vec{v} \neq \vec{0}$$

any vector that satisfies $A\vec{v} = \lambda\vec{v}$ is an eigenvector

$$A\vec{v} = \lambda\vec{v} \implies A\vec{v} - \lambda\vec{v} = \vec{0} \implies (A - \lambda I)\vec{v} = 0$$
$$\implies \vec{v} \in N(A - \lambda I)$$
$$\implies A - \lambda I \text{ is not invertible!}$$
$$\implies \det(A - \lambda I) = 0$$

$\det(A - \lambda I)$ is called the "characteristic polynomial of A"

$\det(A - \lambda I) = 0$ is called the "characteristic equation of A"

☆ Solving the characteristic equation for possible λ's reveals the eigenvalues of A.

$$A := \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad \det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^{2} = (3-\lambda)(3-\lambda)$$

multiplicity 2

$$= 0 \implies \lambda_1 = 3, \ \lambda_2 = 3$$

- We now need to find eigenvectors associated with $\lambda = 3$
  - $N(A - 3I)$
  - but we can find eigenvectors <u>by inspection</u> in this particular case

$$A = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3I$$

Any vector $\vec{x} \in \mathbb{R}^2$ satisfies $A\vec{x} = 3\vec{x}$.

$$\text{Span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}\right)$$

"eigenspace" associated w/ $\lambda = 3$

$$A := \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \qquad \det(A - \lambda I) = (3-\lambda)^2$$

$$N(A - 3I) = N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has to be 0

$$\operatorname{Span}\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right)$$

eigenspace associated with $\lambda = 3$

Fact: distinct eigenvalues produce linearly independent eigenvectors

- but if there are repeated eigenvalues then we have to do a little more work to know how many eigenvectors there are.

Suppose $A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

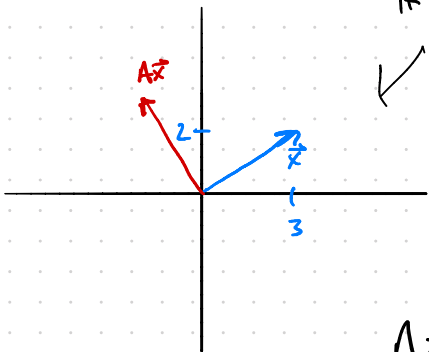
$$\det(A - \lambda I) = (-\lambda)^2 + 1$$

$$= 0 \implies \lambda = \pm\sqrt{-1} = \pm i$$

$:= i$

"imaginary number"

$\mathbb{R}^2$



Exercise:
e-vec associated w/ $-i$ is $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and e-vec associated w/ i is $\begin{bmatrix} i \\ 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

A rotates by $\pi/2$ ...

$$\det(A) = 1 = \prod_i \lambda_i = (-i)(i) = -i^2 = -(-1) = 1 \checkmark$$

general rotation matrix: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Why is this perspective useful in the real world?

- Suppose we have a <u>linear dynamical system</u> with $x[k] = A x[k-1]$ $\leftarrow A \in \mathbb{R}^{N \times N}$

- If A has a set of q linearly independent eigenvectors $v_1, v_2, \ldots, v_q$

... and $x[0] \in \text{Span}(\{v_1, v_2, \ldots, v_q\})$

$$= x[0] = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_q v_q \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_q \in \mathbb{R}$$

Then $x[k] = A^k x[0]$ $\Leftarrow$ $x[1] = A x[0]$, $x[2] = A x[1] = A A x[0] = A^2 x[0]$

$$= A^k (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_q v_q)$$

$$= \alpha_1 \underbrace{A^k v_1} + \alpha_2 A^k v_2 + \ldots + \alpha_q A^k v_q$$

$\swarrow$ These are just scalar!

$$= (\alpha_1 \lambda_1^k) v_1 + (\alpha_2 \lambda_2^k) v_2 + \ldots + (\alpha_q \lambda_q^k) v_q$$

We decomposed $x[k]$ for any k in terms of the eigenvectors and eigenvalues!

Suppose $q = N \leftarrow$ we have a "full set" of eigenvectors

then this analysis (Modal decomposition) applies <u>for any $x[0]$</u>!