MODULE 1 TOPIC REVIEW

TOPICS

- Gaussian Elimination
- Linear Independence & Dependence
- Vector Spaces and Subspaces
- Span vs. Basis
- Column Space vs. Null Space
- Eigenvalues & Eigenvectors
- PageRank
- Derivations and Proofs
- Tips
- Practice exam problems

T

$\vec{x}[0]$

$\vec{x}[1] = T\vec{x}[0]$

$\vec{x}[2] = T\underbrace{\vec{x}[1]}$

$= T(T\vec{x}[0])$

$= T^2\vec{x}[0]$

$\vec{x}[3] = T \cdot \vec{x}[2]$

$= T(T^2\vec{x}[0]$

$= T^3\vec{x}[0]$

$\uparrow T$

$\vec{x}[n] = T^{n}\vec{x}[0]$

GAUSSIAN ELIMINATION

MATRIX TRANSFORMATIONS

- We can view **A**x as a linear operation **A** applied to x
- Geometric operations
  - Rotation
  - Scaling
  - Reflecting
- Order of matrix multiplication matters!

1 a

2 b

3 c

$b(a\vec{x})$

$a\vec{x} \rightarrow \vec{x}_a$

$\underbrace{c \cdot b \cdot a}_{(CBA)\vec{x}} \cdot x$

GAUSSIAN ELIMINATION

- Purpose: solve a linear system of equations for all unknowns.
- How it Works: Build an augmented matrix and perform Gaussian Elimination
- Gaussian Elimination: use row operations to reduce the augmented matrix to reduced row echelon form
    - Scale a row by a scalar value
    - Replace a row with the sum of rows
    - Swap two rows

$$\begin{array}{l} 1. \\ 2. \\ 3. \end{array} \begin{bmatrix} 2x & + & 4y & + & 2z & = & 8 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 4 & 2 & 8 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right]$$

$\vec{R_1} = \alpha R_1$

$\vec{R_2} = 2R_2 + R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

GAUSSIAN ELIMINATION

- Goal: Reduced Form
  - When reduced, you can read the solutions right out of the matrix (if they exist)
- Apply basic row operations to get into this form or reach a stopping condition
- Pivot: Having a nonzero value in a column where all values left of it are 0.
  - In reduced form, all values above and below a pivot are also 0
  - NOTE: Not all matrices will have a pivot in every column

Pivot

Pivot

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

NO Pivot

$1 \cdot x_1 + 0x_2 + 0x_3 = -2$

$\boxed{x_1 = -2}$

$0x_1 + x_2 + 0x_3 = -4$

$\boxed{x_2 = -4}$

$0x_1 + 0x_2 + 0x_3 = 0$

$x_1 = -2$

$x_2 = -4$

$x_3 = ???$

TYPES OF SOLUTIONS

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Unique Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinite Solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No Solution

$0x_1 + 0x_2 + 0x_3 = 1$

LINEAR DEPENDENCE AND INDEPENDENCE

DEFINITIONS

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ $\quad v_3 = -2\vec{v}_1$

$a_1\vec{v}_1 + a_2 v_2 + a_3\vec{v}_3 = 0$

$\boxed{a_1 = 2}$ $a_2 = 0$ $\boxed{a_3 = 1}$

- If a set of vectors is **linearly dependent**, that means at least one vector in the set can be represented as a linear combination of the other vectors in the set.
  - Redundant

$\cancel{2\vec{v}_1} + 0\vec{v}_2 + \cancel{1(-2\vec{v}_1)} = 0$

- If a set of vectors is **linearly independent**, then no vector in that set can be constructed using linear combinations of the other vectors
  - Unique
- To determine if a set of vectors is linearly independent or dependent, solve this equation:

$$\rightarrow \underbrace{a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots a_n\overrightarrow{v_n} = 0}$$

$$a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots a_n\overrightarrow{v_n} = 0 \Rightarrow a_1 = a_2 = \cdots a_n = 0$$

- If all the alphas are 0, then the set of vectors V_1 to V_n is linearly independent. If any alpha is non-zero, then the set of vectors is linearly dependent.

VECTOR SPACES & SUBSPACES

VECTOR SPACE

- Definition: A set of vectors V, scalars F, that satisfy the following vector addition and scalar multiplication operations properties

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w}\in\mathbb{V}$.
  - Commutative: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ for any $\vec{v},\vec{u}\in\mathbb{V}$.
  - Additive Identity: There exists an additive identity $\vec{0}\in\mathbb{V}$ such that $\vec{v}+\vec{0}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$.
  - Additive Inverse: For any $\vec{v}\in\mathbb{V}$, there exists $-\vec{v}\in\mathbb{V}$ such that $\vec{v}+(-\vec{v})=\vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
  - Closure under vector addition: For any two vectors $\vec{v},\vec{u}\in\mathbb{V}$, their sum $\vec{v}+\vec{u}$ must also be in $\mathbb{V}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v})=(\alpha\beta)\vec{v}$ for any $\vec{v}\in\mathbb{V}$, $\alpha,\beta\in\mathbb{R}$.
  - Multiplicative Identity: There exists $1\in\mathbb{R}$ where $1\cdot\vec{v}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$ for any $\alpha\in\mathbb{R}$ and $\vec{u},\vec{v}\in\mathbb{V}$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v}=\alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta\in\mathbb{R}$ and $\vec{v}\in\mathbb{V}$.
  - Closure under scalar multiplication: For any vector $\vec{v}\in\mathbb{V}$ and scalar $\alpha\in\mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{V}$.

VECTOR SUBSPACES

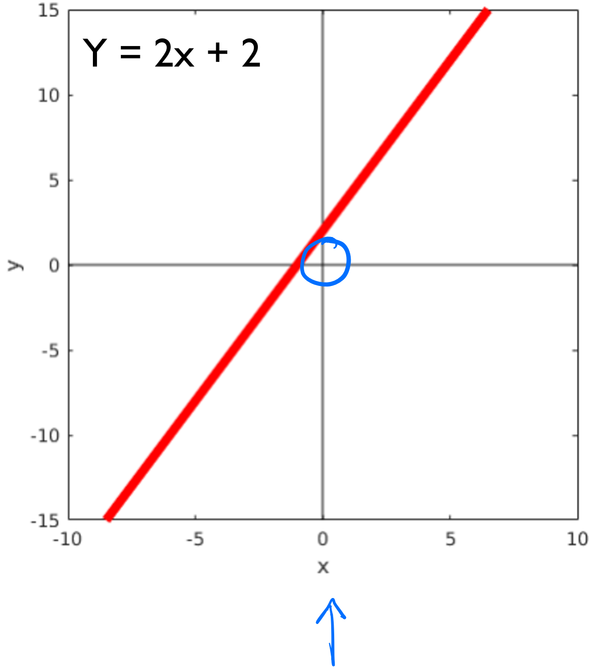
- To verify that a vector space V is a subspace of R^n (or any other vector space) check that it is:
  - A subset of R^n
  - Closed under Vector Addition
  - Closed under Scalar Multiplication
  - Contains the zero vector

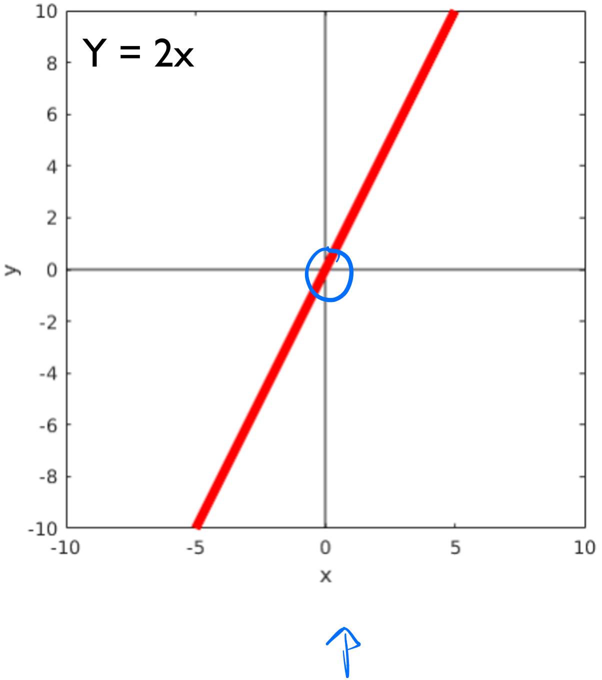
$\vec{x}$, $a\vec{x} \in V$

$a = 0$

$\vec{0} \in V$

ARE THESE SUBSPACES OF V = $\mathbf{R}^2$?





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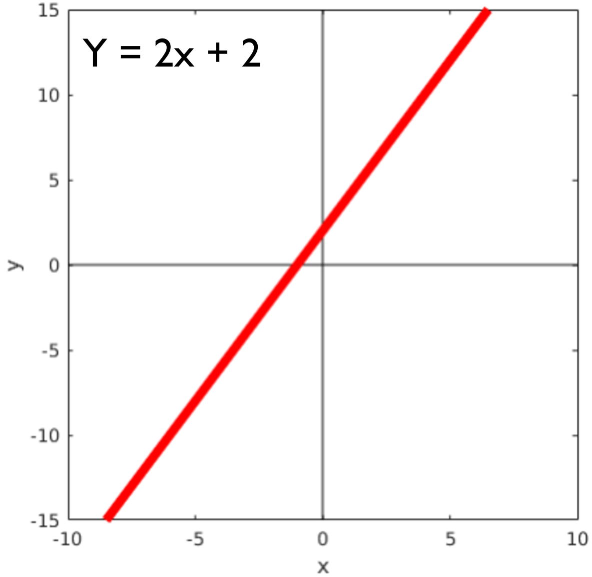
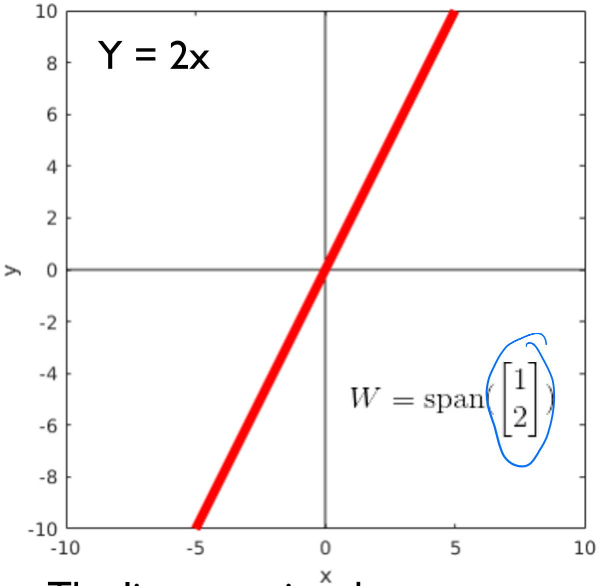


No. Not closed under vector addition or scalar multiplication and doesn't contain the zero vector.



Yes. The line contains the zero vector. We can take linear combinations of points on the line and stay on the line.

DEFINITIONS

- Span (Noun) = All linear combinations of a set of vectors {v1, …, vk}
  - Span({v1, .. vk}) = a1*v1 + … + ak*vk for any scalars a1, …, ak
- Span (Verb) / Spanning (Adj)
  - A list of vectors {v1,..,vk} spans a vector space, V, if every vector in V is in the span({v1,..,vk})
- Basis: a minimum set of vectors that span the space & are linearly independent

DIMENSION

- Dimension: number of vectors in the basis
  - All possible bases for a vector space contain the same number of vectors
  - The number of vectors in the basis is the dimension
- ** Any set of n linearly independent vectors in a n-dimensional vector space is a basis for that space

$\mathbb{R}^n \rightarrow$ n lin. indep.

basis

① addition

$\vec{v}, \vec{w} \in V \quad \vec{v}+\vec{w} \in V$

② scalar multiplication

$\vec{w} \in V \longrightarrow \alpha\vec{w} \in V$

COLUMN SPACE & NULL SPACE

COLUMN SPACE

$\underbrace{f(\vec{x})} = \underbrace{A\vec{x}}$

$\hookrightarrow = \underbrace{x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + \vec{x}_n \vec{a}_n}$

- Let **A** have columns $A_1, \ldots, A_n$.
- Column Space(**A**) = Span($\{A_1, \ldots, A_n\}$)
- Range(**A**) = What you can "reach" with that matrix
  - All **y** such that there exists an x where **A**x = y
  - Dim Range(**A**) = Rank(A)
- These ideas are equivalent:
  - Column Space(**A**) = span($\{A_1, .., A_n\}$) = Range(**A**)

NULL SPACE

$f(\vec{x}) = A\vec{x}$

0

$A\vec{x} = 0$

- Null(**A**) is the set of vectors that map to a zero output:
  - Null(**A**) = {x such that **A**x = 0}
- This is related to the idea of linear independence:
  - If columns of **A** are linearly independent, then Null(**A**) = {0}, i.e. Trivial
  - If columns of **A** are linearly dependent, then the Nul(**A**) is a subspace of dimension 1 or more i.e. Non-Trivial

HOW TO FIND NULL SPACE

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \Rightarrow \begin{matrix} ① \\ ② \\ ③ \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 0x_2 - x_3 = 0 \rightarrow x_1 = x_3$$

$$0x_1 + x_2 + 2x_3 = 0 \rightarrow x_2 = -2x_3$$

$$0 \qquad = 0$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$\text{Null Space}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

WHY DO COLUMN SPACE(**A**) AND NULL(**A**) MATTER?

- We can interpret the solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$
  - Types of Solutions
    - No Solution: if **b** is not in the Col(A)
    - Unique Solution: if b is in the Col(A) and Nul(A) is trivial
    - Infinite Solutions: if b is in the Col(A) but Nul(A) is non-trivial

$\vec{y} \in Nul(A)$

$\vec{x} = (\vec{z} + \vec{y})$

$A\vec{x} = b$

$A\vec{z} = \vec{b}$

$A(\vec{z} + \vec{y}) = \vec{b}$

$A\vec{z} + \cancel{A\vec{y}} = \vec{b}$

$\downarrow$

$\vec{0}$

$A\vec{z} = \vec{b}$

$\vec{b} \in Col(A)$ | $\vec{b} \notin Col(A)$

Nul(A) trivial? | $\downarrow$ no solution

unique | infinite

WHY DO COLUMN SPACE(**A**) AND NULL(**A**) MATTER?

- We can interpret the solutions to A**x** = **b**
  - Invertibility
    - If & only If Nul(A) is trivial
  - Types of Solutions
    - No Solution: if **b** is not in the Col(A)
    - Unique Solution: if b is in the Col(A) and Nul(A) is trivial
    - Infinite Solutions: if b is in the Col(A) but Nul(A) is non-trivial

INVERTIBILITY

- Only square matrices can be invertible
- **A** (n x n) is not invertible if it
  - it has linearly dependent rows (or columns)
  - has a non-trivial null space (a null space that doesn't just contain zero)
  - has a zero eigenvalue (more on eigenvalues later)
  - has zero determinant (more on determinants later)
- If any of the statements in the above list are false, then **A** is invertible.

invertible matrix theorem

DETERMINANTS

- For this class, you need to know 2x2 determinants

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

- Determinants encode some very neat info about **square** matrices
- They are linked to the area (2D) of the parallelogram or volume (3D) or parallelepiped formed by the rows (or cols) of the **A** matrix
- Important for finding eigenvalues of a matrix **A**
- det(**A**) = 0 → non-empty Null(**A**)

EIGENVALUES & EIGENVECTORS

EIGENVECTORS/EIGENVALUES (AND EIGENSPACES)

- Eigenvector: a **non-zero** vector such that applying matrix **A** to it yields the **SAME** exact vector, but scaled by a constant.
- Eigenvalue: the constant that the eigenvector is being scaled by
- Eigenspace: the span of all the eigenvectors associated with a singular eigenvalue

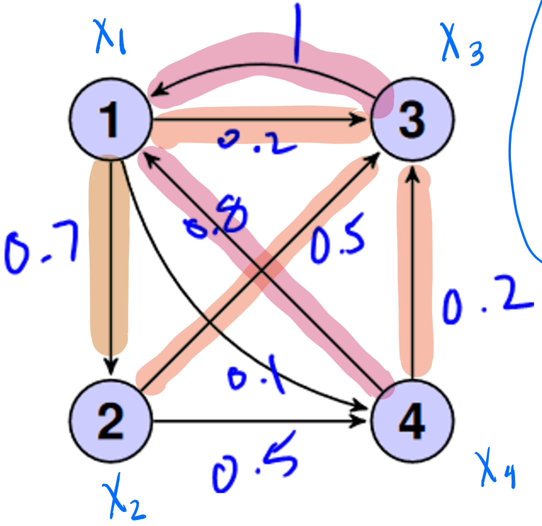
$$A\vec{v} = \lambda\vec{v}$$

PAGERANK

FLOW MATRICES AND GRAPHS

- A system that updates its state every "time step"
- Usually represented by nodes connected by directed edges with some "weight"
  - Weight can mean different things depending on the system.
    - Water tanks → percentage of water that goes to a different (or the same) tank
    - Pagerank → percentage of people that goes to a different webpage

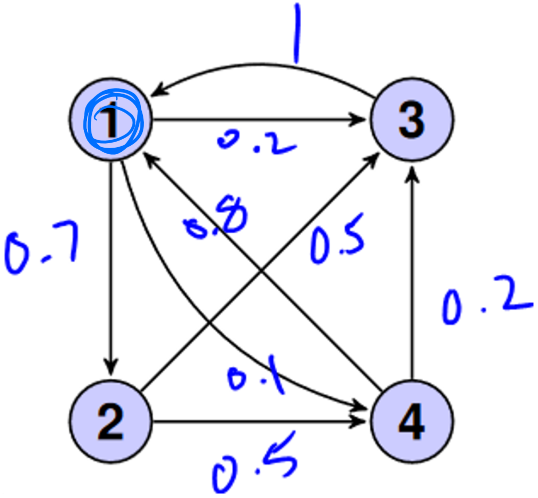
TRANSITION MATRICES



$$\left\{\begin{aligned}
X_1[t+1] &= 0\,X_1[t] + 0\,X_2[t] + X_3[t] + 0.8X_4[t] \\
X_2[t+1] &= 0.7\,X_1[t] + 0 + 0 + 0 \\
X_3[t+1] &= 0.2X_1[t] + 0.5X_2[t] + 0 + 0.2X_4[t] \\
X_4[t+1] &= 0.1\,X_1[t] + 0.5X_2[t]
\end{aligned}\right.$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0.8 \\ 0.7 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 \\ 0.1 & 0.5 & 0 & 0 \end{bmatrix}}\begin{bmatrix} X_1[t] \\ X_2[t] \\ X_3[t] \\ X_4[t] \end{bmatrix} = \begin{bmatrix} X_1[t+1] \\ \vdots \\ \vdots \end{bmatrix}$$

TRANSITION MATRICES



Node 1 to all other nodes

$$T = \begin{bmatrix} 0 & 0 & 1 & 0.8 \\ 0.7 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 \\ 0.1 & 0.5 & 0 & 0 \end{bmatrix}$$

All other nodes to node 1

FLOW MATRICES AND GRAPHS

- Conservative System: nothing leaves or enters the system as time progresses.
  - When all the columns sum to 1
  - A column with sum >1 has "stuff" entering from outside. NOT conserved, states go to infinity
  - A column with sum < 1 has "stuff" leaking out. States will trend towards 0.
- State transitions?
  - We have a state as a vector s[n]
  - Transition matrix **T** describes how the state changes at each time step.

$$T\vec{s}[n] = \vec{s}[n+1]$$

FLOW MATRICES AND GRAPHS

- Next states? Previous states?
  - We have a state as a vector s[n] which represents how much stuff is at which node
  - Transition matrix **T** describes how the state changes at each time step.

$$T\vec{s}[n] = \vec{s}[n+1]$$

  - We can advance many timesteps by computing powers of **T**

$$T^N\vec{s}[n] = \vec{s}[n+N]$$

  - We can go backward in time to recover previous states IF **T** IS INVERTIBLE

$$T^{-1}\vec{s}[n+1] = \vec{s}[n]$$

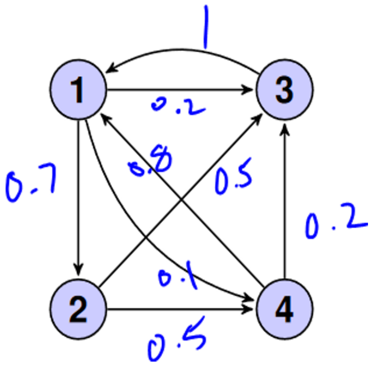
STEADY STATE

- We're interested in a state where updating to the next step does not change the state vector
  - In other words… $1 = \lambda$

$$T\vec{s}[n] = \vec{s}[n+1] = \vec{s}[n]$$

  - Maybe it looks more familiar written this way…

$$A\vec{v} = \vec{v}$$



$$T = \begin{bmatrix} 0 & 0 & 1 & 0.8 \\ 0.7 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 \\ 0.1 & 0.5 & 0 & 0 \end{bmatrix}$$

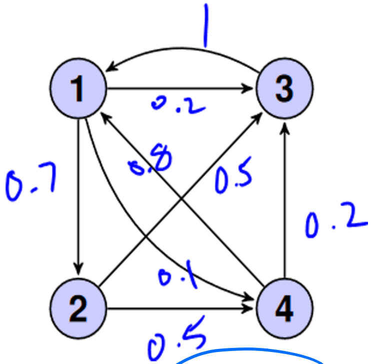
- There's just one last thing to do...

- There's an eigenvalue of 1. Thus a steady state exists

- The corresponding eigenvector is:

$$\rightarrow \quad \underline{\underline{\overrightarrow{v_1}}} = \begin{bmatrix} 0.6897 \\ 0.482 \\ 0.441 \\ 0.3103 \end{bmatrix}$$

1000



$$T = \begin{bmatrix} 0 & 0 & 1 & 0.8 \\ 0.7 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 \\ 0.1 & 0.5 & 0 & 0 \end{bmatrix}$$

$$\overrightarrow{v_1} = \begin{bmatrix} 0.6897 \\ 0.482 \\ 0.441 \\ 0.3103 \end{bmatrix} \rightarrow \begin{bmatrix} 0.3584 \\ 0.2508 \\ 0.2293 \\ 0.1612 \end{bmatrix}$$

358.4
250.8
229.3

Normalize to get percentages of population!!

- These numbers don't make sense for our system. We need to "normalize" it so that the total sums to 1, representing 100%
- Divide each entry by the sum of all values
- Therefore at steady state, 35.84% of the stuff is in node 1, 25.08% in node 2, etc.

a) If the augmented matrix of the system Ax = b has a pivot in the last column, then the system Ax = b has no solution.
b) If A and B are invertible 2×2 matrices, then $(AB)^{-1} = A^{-1}B^{-1}$
c) If A is a 3×3 matrix such that the system A**x** = 0 has only the trivial solution, then the equation A**x** = **b** is consistent for every **b** in R^3 .
d) If T $[x\ y]^T = [x\ 0]^T$, then Nul(T) = span$\{[1, 0]^T\}$
e) R^2 is a subspace of R^3
f) If Nul(A) = {0}, then A is invertible.
g) If {**v1, v2, v3**} are linearly independent vectors in R^n , then {**v1, v2**} is linearly independent as well.

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Spring '19 Question 4b

Let $\mathbf{U}$ and $\mathbf{V}$ be $n \times n$ matrices. If $\mathbf{UV} = \mathbf{0}$, prove that every vector in $\mathrm{col}(\mathbf{V})$ is in $\mathrm{nul}(\mathbf{U})$.

6. A Tropical Tale of Triumph: Does Pineapple Come Out on Top? (52 points)

(Based on a true story) During a discussion section, one of your TAs, Nick, makes the claim that pineapple belongs on pizza. Another TA, Elena, strongly disagrees. Naturally, a war starts and students begin to flock to the TA they agree with, switching discussion sections every week. Some students don't have an opinion and go to Lydia's section since she is neutral in the matter. As a 16A student, you want to analyze this war to see how it will play out.

(a) (6 points) You manage to capture the behavior of the students as a transition matrix, but want to visualize it. You've written out the transition matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.25 & 0.5 & 1 \\ 0.25 & 0.5 & 0 \end{bmatrix}$$

such that

$$\begin{bmatrix} x_{\text{Elena}}[n+1] \\ x_{\text{Nick}}[n+1] \\ x_{\text{Lydia}}[n+1] \end{bmatrix} = \mathbf{M} \begin{bmatrix} x_{\text{Elena}}[n] \\ x_{\text{Nick}}[n] \\ x_{\text{Lydia}}[n] \end{bmatrix}.$$

(b) (10 points) Your friend Vlad tells you that your transition matrix $\mathbf{M}$ was wrong, and gives you a new transition matrix $\mathbf{S}$, **which has a steady state**. In order to find who wins the war, you need to find how many students end up in each section after everything has settled. **Find a vector $\vec{x}$ that represents a steady state of $\mathbf{S}$.**

$$\mathbf{S} = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 1 \end{bmatrix}$$

(c) (6 points) Your other friend Gireeja points out that the arguments are causing new people to join the sections and others to leave entirely. In other words, the system is not conservative! The new system can be modeled with a state transition matrix **A** that has the following eigenvalue/eigenvector pairings:

$$\lambda_1 = 1 : \vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} : \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2 : \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You want the number of students in sections to stabilize. Which of the vectors below represent **steady states** of the system, i.e. $\vec{x}$ such that $\mathbf{A}\vec{x} = \vec{x}$? **Fill in the circle(s) to the left of these vector(s).**

$\bigcirc \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ $\bigcirc \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\bigcirc \begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ $\bigcirc \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

$\bigcirc \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\bigcirc \begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ $\bigcirc \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\bigcirc \begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$

(d) (6 Points) Assume we are still working with the same state transition matrix **A** as in part (c). Which of the vectors below represent **initial states** such that the number of students in the sections keeps growing? **Fill in the circle(s) to the left of these vector(s).**

○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$

(e) (6 points) Again assume we are still working with the same state transition matrix $\mathbf{A}$ as in part (c). Which of the vectors below represent **initial states** such that everyone leaves the system, i.e. $\lim_{n\to\infty} \mathbf{A}^n \vec{x} = \vec{0}$? **Fill in the circle(s) to the left of these vector(s).**

○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$

(f) (16 Points) Let us generalize the idea of convergence. Consider the following system:

$$\vec{x}[n+1] = \mathbf{T}\vec{x}[n]$$

where $\vec{x}$ is a vector with N elements and $\mathbf{T}$ is any $N \times N$ matrix unrelated to the previous parts. $\mathbf{T}$ has N distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, and N associated eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$ such that $\mathbf{T}\vec{v}_i = \lambda_i \vec{v}_i$ for $1 \leq i \leq N$. Let $|\lambda_i| > 1$. Prove that there exists at least one initial state $\vec{x}[0]$ for this system such that it does not converge to a steady state.

FROM FALL 2016 MT I

6. Directional Shovels (10 points)

Kody and Nara were found exceptional at taking measurements to figure out light intensities, and they were both granted admission to a graduate school. Unfortunately, they both supported their new school's football team while they were playing against Berkeley and angry Berkeley fans found them and left them in a room at an unknown location under the ground. As compassionate people, Berkeley fans left some tools in the room that can help them escape.

(a) Kody found a shovel in the room and figured that it can operate in the following directions: $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$. Is it possible for them to escape to Berkeley by digging in the given directions to a point which is located at $\begin{bmatrix}3\\-2\\5\end{bmatrix}$ given that they are at point $\begin{bmatrix}0\\0\\0\end{bmatrix}$? If so, find the scalars that multiply the vectors such that they reach Berkeley.

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(b) While Kody was busy planning his escape to Berkeley, Nara found a pick-axe in the room that can operate in the following directions: $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$. Nara is convinced that the axe she found is better, but Kody disagrees. Show that Kody's shovel can reach anywhere that Nara's pick-axe can.

(b) While Kody was busy planning his escape to Berkeley, Nara found a pick-axe in the room that can operate in the following directions: $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right\}$. Nara is convinced that the axe she found is better, but Kody disagrees. Show that Kody's shovel can reach anywhere that Nara's pick-axe can.

FALL 2019 MT1

3. Campfire Smores (11 points)

Patrick and SpongeBob are making smores.

There are three ingredients: **Graham Crackers, Marshmallows, and Chocolate**. To make a smore, SpongeBob needs: s_g Graham Crackers, s_m number of Marshmallows, and s_c Chocolate.

Ingredients	**Amount Needed**
Graham Crackers (s_g)	10
Marshmallows (s_m)	14
Chocolate (s_c)	20

Table 3.1: SpongeBob's smore

They find out that these ingredients are only stored in bundles as below:

Lobster Pack (p_l)
6 graham crackers
4 marshmallows
2 chocolates

Mr. Krabs Pack (p_k)
2 graham crackers
2 marshmallows
1 chocolates

Squidward Pack (p_s)
3 graham crackers
3 marshmallows
5 chocolates

Gary Pack (p_g)
1 graham crackers
4 marshmallows
5 chocolates

Pearl Pack (p_p)
2 graham crackers
3 marshmallows
2 chocolates

Table 3.2: Amount of Ingredients per Bundle

Spongebob and Patrick need to know how many of each bundle to buy: number of "Lobster" Packs, p_l, number of "Mr. Krabs" Packs, p_k, number of "Squidward" Packs, p_s, number of "Gary" Packs, p_g, and number of "Pearl" Packs, p_p.

(a) (3 points) How many equations/constraints does the information in the problem provide you with?

(b) (4 points) Based on the information provided in Tables 3.1 and 3.2, **write** an equation of the form $\mathbf{A}\vec{p} = \vec{s}$ that SpongeBob can use to decide how many of each pack to buy. Here, $\vec{p} = \begin{bmatrix} p_l \\ p_k \\ p_s \\ p_g \\ p_p \end{bmatrix}$.

(c) (4 points) Now, the ingredients in the packets ($\mathbf{A}$) and Spongebob's receipe ($\vec{s}$) change. We have:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0 & 2 \\ 1 & 3 & 9 & 2 & 6 \end{bmatrix}, \text{ and } \vec{s} = \begin{bmatrix} 3 \\ 2 \\ 10 \end{bmatrix}.$$

Find a $\vec{p}$ that satisfies $\mathbf{A}\vec{p} = \vec{s}$. If no solution exists, explain why not.