EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 2D

1. Aragorn's Odyssey

In a desperate attempt to save Minas Tirith, Aragorn is trying to maneuver your ship in a 2D plane around the fleet of the Corsairs of the South. The position of your ship in two dimensions (x, y) is represented as a vector, $\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) In order to evade the Witch-King of Angmar, Gandalf provides Aragorn with linear transformation spell. The spell first reflects your ship along the X-axis (i.e. multiplies the Y-coordinate by -1) and then rotates it by 30 degrees counterclockwise. Express the transformation Gandalf's spell performed on the ship's location as a 2×2 matrix.

Hint: Recall that the matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *rotates a vector counterclockwise by* θ.

Answer:

$$\begin{aligned} \mathbf{G_{spell}} &= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \end{aligned}$$

(b) If the ship was initially 1 unit distance away from the origin $(0, 0)$, how far is it from the origin after the transformation above? Justify your answer.

Answer: 1 unit still - the transformation should not affect the distance from the origin. This can be explained geometrically in terms of what rotations and reflections represent (operations that do not change the distance from the origin). TAs may also explore a brief aside about the determinants of rotation and reflection matrices and how this relates to the distance-preserving property

(c) Having evaded the Witch-King and the Corsairs, Aragorn needs to quickly reach Minas Tirith. To do so, he uses the wind spell, $\mathbf{B}_{\text{spell}}$, ten times, where his position $\vec{x}[t]$ changes according to the equation

$$\vec{x}[t+1] = \mathbf{B}_{\text{spell}} \vec{x}[t],$$

where $\mathbf{B}_{\text{spell}} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$.

The initial location of your ship is $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **What is the location of your ship at time $t = 10$, i.e. what is $\vec{x}[10]$?** Explicitly compute your final solution and justify your answer.

Answer: Rather than directly computing $\mathbf{B}_{\text{spell}}^{10}$ we should be suspicious that a problem like this will involve eigenvalues and eigenvectors, which make the situtation much more simple. If we compute the eigenvalues and eigenvectors of $\mathbf{B}_{\text{spell}}$ we find that we have $\lambda_1 = 2, \lambda = 3$ with eigenvector $\vec{v}_1 = [1, 0]^T$ corresponding to eigenvalue $\lambda_1 = 2$. Therefore the given initial condition $\vec{x}[0]$ is an eigenvector of the matrix $\mathbf{B}_{\text{spell}}$! As a result, at time 10 we simply have: $\vec{x}[10] = 2^{10} \vec{x}[0]$

(d) The ship is now moving in an n dimensional space. The position of the ship at time t is represented by $\vec{x}[t] \in \mathbb{R}^n$. The ship starts at the origin $\vec{0}$.

Aragorn tries a new spell, $\mathbf{C}_{\text{spell}} \in \mathbb{R}^{n\times n}$, $\mathbf{C}_{\text{spell}} \neq 0$. In addition to the spell, the ship is given some ability to steer using the scalar input $u[t] \in \mathbb{R}$. The location of the ship at the next time step is described by the equation:

$$\vec{x}[t+1] = \mathbf{C}_{\text{spell}}\vec{x}[t] + \vec{b}u[t],$$

where $\vec{b} \in \mathbb{R}^n$ is fixed.

You know from the Segway problem on the homework that the ship can reach all locations in the $\text{span}\{\vec{b}, \mathbf{C}_{\text{spell}}\vec{b}, \mathbf{C}^2_{\text{spell}}\vec{b}, \cdots, \mathbf{C}^9_{\text{spell}}\vec{b}\}$ in ten time steps. **Suppose we tell you that $\vec{b} \neq 0$ is an eigenvector of $\mathbf{C}_{\text{spell}}$.** What is the maximum dimension of the subspace of locations the ship can reach? Justify your answer.

Answer: Since $\vec{b}$ is an eigenvector of $\mathbf{C}_{\textbf{spell}}$ we will have $\mathbf{C}^{\mathbf{k}}\vec{b} = \lambda_b^k\vec{b}$, this means that:
$\dim(\text{span}\{\vec{b}, \mathbf{C}_{\text{spell}}\vec{b}, \mathbf{C}^2_{\text{spell}}\vec{b}, \cdots, \mathbf{C}^9_{\text{spell}}\vec{b}\}) = 1$

2. Trouble in Telecomm

Fred (x_0), Tina (x_1), and Will (x_2) each are sending messages (where each message x_0, x_1, x_2 is a real number) at the same time to Alec, Kristin, and Colin respectively.

To achieve this, the phone company will transmit $\vec{y}$, which is a vector of linear combinations of x_0, x_1, x_2. Specifically,

$$\vec{y} = \mathbf{V}\vec{x} = \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}. \tag{1}$$

$\mathbf{V}$ is the encoding matrix.

On the receiver side, Alec, Kristin and Colin need to recover x_0, x_1, x_2 respectively from $\vec{y}$. You are helping the phone company evaluate different choices for the columns $\vec{c}_0$, $\vec{c}_1$ and $\vec{c}_2$ of matrix $\mathbf{V}$:

$$\begin{aligned} \mathbf{V_0} &= \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix} \\ \mathbf{V_1} &= \begin{bmatrix} | & | & | \\ \vec{c}_0 & \vec{c}_1 & \vec{c}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{aligned} \tag{2}$$

(a) You decide to characterize $\mathbf{V_0}$ in terms of its null space. Find a basis for the nullspace of $\mathbf{V_0}$.

Answer: Analyzing the first set of codes, we can find the null space by setting up the following matrix equation:

$$\mathbf{V}_0\vec{x} = \vec{0} \tag{3}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \tag{4}$$

$$\overbrace{\Rightarrow}^{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \tag{5}$$

$$\overbrace{\Rightarrow}^{R_2/3 \to R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \tag{6}$$

$$\overbrace{\Rightarrow}^{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{0} \tag{7}$$

Now we identify c is a free variable and express the other variables in terms of it.

$$b = -2c \tag{8}$$
$$a = -2c \tag{9}$$

Finally we construct the null space,

$$N(\mathbf{V}_0) = \text{span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\} \tag{10}$$

(b) If the matrix $\mathbf{V_0} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 2 & 4 \end{bmatrix}$ is invertible, find its inverse. If it is not invertible, why not? Given this, is $\mathbf{V_0}$ a good encoding matrix to use? Justify your answer.
Answer: The matrix $\mathbf{V_0}$ is non-invertible. This is because the columns are not linearly independent (twice the second plus the first is the third). This can also be identified from the null space. The null space of the matrix contains more than just the zero vector. This implies there will be an infinite number of solutions and that Alec, Kristin, and Colin will not be able to uniquely decode the sent messages. This makes $\mathbf{V_0}$ an unsuitable encoding matrix for the telecommunication company.

(c) If the matrix $\mathbf{V_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible, find its inverse. If it is not invertible, why not? Given this, is $\mathbf{V_1}$ a good encoding matrix to use? Justify your answer.
Answer: The columns are linearly independent and therefore the matrix inverse exists. To find the inverse, we start by writing the matrix equation:

$$\mathbf{V}_1\mathbf{V}_1^{-1} = \mathbf{I} \tag{11}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{V}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Then we setup the augmented matrix:

$$\left[\begin{array}{c|c} \mathbf{V}_1 & \mathbf{I} \end{array}\right] \tag{13}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{14}$$

$$\underbrace{\Rightarrow}^{\substack{R_3 \to R_1 \\ R_1 \to R_2 \\ R_2 \to R_3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \tag{15}$$

$$\underbrace{\Rightarrow}^{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \end{array}\right] \tag{16}$$

$$\underbrace{\Rightarrow}^{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right] \tag{17}$$

$$\underbrace{\Rightarrow}^{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right] \tag{18}$$

$$\underbrace{\Rightarrow}^{R_2 - R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right] \tag{19}$$

$$\underbrace{\Rightarrow}^{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{array}\right] \tag{20}$$

Reading off the right half of the augemented matrix, we get:

$$\mathbf{V}_1^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \tag{21}$$

Because the inverse exists, Alec, Kristin, and Colin will be able to uniquely decode their intended signals, thus, $\mathbf{V}_1$ is a suitable encoding matrix for the telecommunication company.

3. Free-form review with discussion section TAs (if time)