EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 2C

1. Steady and Unsteady States

(a) You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$. ($\vec{x}$ could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Answer:

i. $\lambda = 1$:

$$\left[\begin{array}{c|c} \mathbf{M}-\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

ii. $\lambda = 2$

$$\left[\begin{array}{c|c} \mathbf{M}-2\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} \frac{-3}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \beta \in \mathbb{R}$$

iii. $\lambda = \frac{1}{2}$

$$\left[\begin{array}{c|c} \mathbf{M}-\frac{1}{2}\mathbf{I} & \vec{0} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \gamma \in \mathbb{R}$$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty}\mathbf{M}^n\vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

Answer:

$$\begin{aligned}\mathbf{M}^n\vec{x} &= \mathbf{M}^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3)\\ &= \alpha\mathbf{M}^n\vec{v}_1 + \beta\mathbf{M}^n\vec{v}_2 + \gamma\mathbf{M}^n\vec{v}_3\\ &= 1^n\alpha\vec{v}_1 + 2^n\beta\vec{v}_2 + \left(\frac{1}{2}\right)^n\gamma\vec{v}_3\end{aligned}$$

α	β	γ	Converges?	$\lim_{n\to\infty}\mathbf{M}^n\vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	-
0	$\neq 0$	$\neq 0$	No	-
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	-
$\neq 0$	$\neq 0$	$\neq 0$	No	-

2. Polynomials as a Vector Space

Let $\mathbb{P}_2$ be the set of polynomials of degree of at most two (that is, $p(t) = at^2 + bt + c$).

(a) Give a basis for $\mathbb{P}_2$.

Answer:

The standard basis would be $\{t^2, t, 1\}$, but any three linearly independent quadratics, where {constant, linear, quadratic} terms all appear at least once, is acceptable.

(b) Consider the linear transformations

$$T_1(f(t)) = 2f(t)$$
$$T_2(f(t)) = f'(t)$$

For each, find the transformation matrix with respect to the basis from part (a).

Answer:

Using the ordered standard basis $\{t^2, t, 1\}$,

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) Suppose that $\{x_0, x_1, x_2\}$ form a basis for $\mathbb{P}_2$ and that the following polynomials have the corresponding coordinates in this basis.

$$\begin{aligned} (1,1,1) &\Rightarrow 2t^2 + 3t \\ (1,0,-1) &\Rightarrow t+1 \\ (0,2,0) &\Rightarrow 4t+2 \end{aligned}$$

Find the basis vectors x_0, x_1, x_2.

Answer:

Solving the system of equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t^2 + 3t \\ t+1 \\ 4t+2 \end{bmatrix}$$

yields the solution:

$$x_0 = t^2 + t, x_1 = 2t + 1, x_2 = t^2 - 1$$