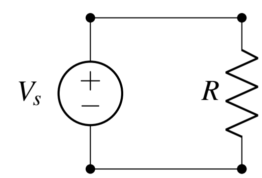
EECS 16A Designing Information Devices and Systems I
Spring 2022 Lecture Notes Note 11B

11.1 Node Voltage Analysis

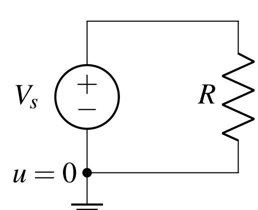
In this course, we will learn how to take a real world system and build a circuit diagram that models the behavior of that system, and we will design our own circuits for specific real world tasks. In this note, however, we will assume that we already have an accurate circuit diagram, and will learn how to analyze the circuit.

For a given circuit, we would like to find all of the voltages and currents – sometimes we call this "solving" the circuit. We'll go through an example using the following diagram, which consists of four elements: a voltage source, a resistor, and two wires.

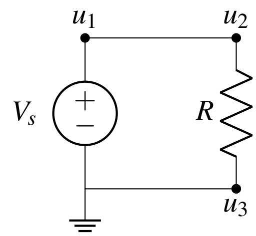


For the sake of clarity, after each step of the analysis algorithm we show what the current circuit diagram looks like. When you perform the algorithm on your own, however, you do not need to redraw the circuit each time; instead you can simply label/annotate a single diagram.

- **Step 1:** Pick a junction and label it as $u = 0$ ("ground"), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.

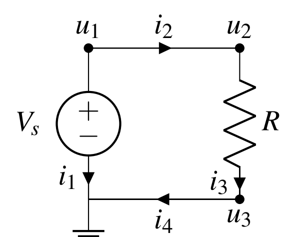


- **Step 2:** Label all remaining junctions as some "u_i", representing the voltage at each junction relative to the zero junction/ground.



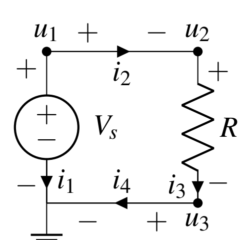
You will see soon that we can simplify our procedure by labeling nodes rather than junctions in the circuit. Once you have some familiarity with the procedure there are simplifications (See 11.2) we can make to avoid analysis in every single wire, but we describe here the most complete and rigorous version.

- **Step 3:** Label the current through every element in the circuit "i_n". Every element in the circuit that was listed above should have a current label, including ideal wires. The direction of the arrow indidates which direction of current flow you are considering to be positive. At this stage of the algorithm, you can pick the direction of all of the current arrows *arbitrarily* - as long as you are consistent with this choice and follow the rules described in the rest of this algorithm, the math will work out correctly.



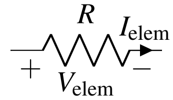
Note that we only label the current once for each element – for example, we can label i_3 as the current leaving the resistor (as is done in the diagram) *or* we can label it as the the current entering the resistor. These are equivalent because KCL also holds within the element itself – i.e., the current that enters an element must be equal to the current that exits that same element.

- **Step 4:** Add $+/-$ labels on each element, following **Passive Sign Convention** (discussed below). These labels will indicate the direction with which voltage will be measured across that element.



Passive sign convention

The **passive sign convention** dictates that positive current should *enter* the positive terminal and *exit* the negative terminal of an element. Below is an example for a resistor:



As long as this convention is followed consistently, it does not matter which direction you arbitrarily assigned each element current to; the voltage referencing will work out to determine the correct final sign. When we discuss *power* later in the module, you will see why we call this convention "passive."

At this stage in the circuit analysis algorithm, we find that there are several **unknowns** labelled on our circuit. These are: $i_1, i_2, i_3, i_4, u_1, u_2, u_3$.

- **Step 5:** Use KCL to write equations with our unknowns.

  Let's begin by writing KCL equations for every junction in the circuit.

$$\begin{aligned} i_1 + i_2 &= 0 \\ -i_2 + i_3 &= 0 \\ -i_3 + i_4 &= 0 \\ -i_4 - i_1 &= 0 \end{aligned}$$

  Notice the last equation we get is linearly dependent with the first three - you can see this by adding all three of the first equations to each other and multiplying the entire result by -1. We will therefore omit this equation. Note that in general, if you use KCL at every junction, you will get one linearly dependent equation, and so you can typically simply skip one junction; skipping the junction that has been labeled as ground is a common choice.

- **Step 6:** Use the IV relationships of each of the elements.

  We know that the difference in potentials across the voltage source must be the voltage on the voltage source. We also know that the voltage across the resistor is equal to the current times the resistance, from Ohm's Law. For the wires, we know the difference in potential is 0. Thus, we have the following equations:

$$\begin{aligned} u_1 - 0 &= V_s \\ u_1 - u_2 &= 0 \\ u_2 - u_3 &= Ri_3 \\ u_3 - 0 &= 0 \end{aligned}$$

  Since u_3 a junction connected to ground, u_3 is simply 0. Again, this shows that it is not always necessary to label all junctions (see 11.2 below).

- **Step 7:** Simplify your equations and solve. Let's take a look at all our equations.

$$
\begin{aligned}
i_1 + i_2 &= 0 \\
-i_2 + i_3 &= 0 \\
-i_3 + i_4 &= 0 \\
-i_4 - i_1 &= 0 \\
u_1 - 0 &= V_s \\
u_1 - u_2 &= 0 \\
u_2 - u_3 &= Ri_3 \\
u_3 - 0 &= 0
\end{aligned}
$$

Recall that we noticed the fourth equation is linearly dependent with the first three, so we will omit this equation since it will not present any new information to solve our system. Simplifying our equations, we now have:

$$
\begin{aligned}
i_1 + i_2 &= 0 \\
-i_2 + i_3 &= 0 \\
-i_3 + i_4 &= 0 \\
u_1 &= V_s \\
u_1 &= u_2 \\
u_2 - u_3 &= Ri_3 \\
u_3 &= 0
\end{aligned}
$$

Simplifying the last four equations, we get:

$$
\begin{aligned}
u_1 &= V_s \\
u_2 &= V_s \\
u_2 &= Ri_3 \\
u_3 &= 0
\end{aligned}
$$

These values make sense. u_1 and u_2 are both connected to V_s by a wire, thus $u_1 = u_2 = V_s$. The junction u_3 is connected to ground by a wire, thus u_3 = 0. Finally, we can find $i_3 = u_2/R = V_s/R$.

Knowing i_3 we can use substitution and find that $i_4 = i_2 = -i_1 = V_s/R$. Using substitution, we have found all uknowns.

Note that at this step, if your system of equations is too complex, you may also choose to use a different method of finding solutions to your unknowns (discused in the next section).

Will we always have as many equations as we do unknowns? If a circuit has m elements in it and n junctions, there will be $(n-1)$ u's (since we have defined one of them as ground/zero), and m currents (one for each element). Since each element has a defining I-V relationship, Step 6 will provide us with m equations. Similarly, with n junctions, we will get $(n-1)$ linearly independent KCL equations from Step 5. This holds true if we use label nodes instead of junctions (see 11.2).

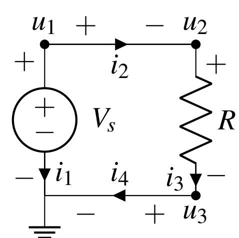
Circuit Analysis with Matrices

Alternatively, we can perform circuit analysis using linear algebra techniques we have learned. Note that this approach is essentially the same as the steps presented previously. However, here we are approaching the problem with the intent of using matrices to analyze our circuit. In more complex circuits, this method will be very useful (see 11.3 Example).

- **Goal:** The goal is to set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the unknown circuit variables we want to solve for (currents and node potentials – that is, the i's and u's). $\mathbf{A}$ will be an $n \times n$ matrix where n is equal to the number of unknown variables. For the circuit above, we have 3 unknown potentials (u) and 4 unknown currents (i), therefore we form a 7×7 matrix.

$$\begin{bmatrix} ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

Here is the circuit again after Step 4.



- **Alternative Step 5:** Use KCL to fill in as many **Linearly Independent** rows of $\mathbf{A}$ and $\vec{b}$ as possible.

Let's begin by writing KCL equations for every junction in the circuit.

$$\begin{aligned} i_1 + i_2 &= 0 \\ -i_2 + i_3 &= 0 \\ -i_3 + i_4 &= 0 \\ -i_4 - i_1 &= 0 \end{aligned}$$

Notice the last equation we get is linearly dependent with the first three - you can see this by adding all three of the first equations to each other and multiplying the entire result by -1. In order to end up with a square and invertible $\mathbf{A}$ matrix, we will therefore omit this equation. Note that in general, if you use KCL at every junction, you will get one linearly dependent equation, and so you can typically simply skip one junction; skipping the junction that has been labeled as ground is a common choice.

Now we put these equations in matrix form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

- **Alternative Step 6:** Use the IV relationships of each of the elements to fill in the remaining equations (rows of **A** and values of $\vec{b}$).

  In this example, we need four more linearly independent equations, and there are four circuit elements, each with their own IV relationship (this is not a coincidence, as will be explained shortly). We use what we know about each element to form four more equations.

  We know that the difference in potentials across the voltage source must be the voltage on the voltage source. We also know that the voltage across the resistor is equal to the current times the resistance, from Ohm's Law. For the wires, we know the difference in potential is 0. Thus, we have the following equations:

$$\begin{aligned} u_1 - 0 &= V_s \\ u_1 - u_2 &= 0 \\ u_2 - u_3 &= R i_3 \\ u_3 - 0 &= 0 \end{aligned}$$

  After filling in these equations, our matrix is:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -R & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- **Step 7:** Solve.

  At this point the analysis procedure is effectively complete - all that's left to do is solve the system of linear equations (by applying Gaussian Elimination, inverting **A**, computationally, etc.) to find the values for the u's and i's.

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -V_s/R \\ V_s/R \\ V_s/R \\ V_s/R \\ V_s \\ V_s \\ 0 \end{bmatrix}$$

11.2 Simplifying the Circuit Analysis Procedure

While the analysis procedure we described in the previous section will always work, and introducing the procedure at this level of comprehensiveness is necessary to ensure that one can always follow it successfully, as is most likely clear, even for very simple circuits the procedure will quickly involve a large number of variables and hence large matrices. Fortunately, we can substantially reduce the number of variables by noticing two things:

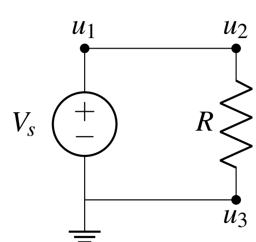
1. There is no voltage drop across wires. Therefore, the node potentials at two ends of a wire are always equal.
2. When a junction involves only two elements, KCL tells us that the current flowing in through the first element must equal the current flowing out through the second element.

The next two sections describe in more detail how we can use these observations to simplify solving a circuit.

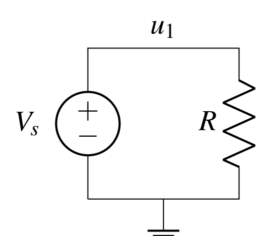
11.2.1 Labeling Nodes Instead of Junctions

Since wires always have zero voltage drop across them, there is no specific need for us to keep track of the voltage (relative to ground) on the two sides of a wire separately. In other words, **all of the junctions that are connected to each other by wires can be labeled with a single voltage variable u.** A set of such junctions connected to each other only via wires is defined as a **node**. (Formally, a node is defined as a region of the circuit that is "equipotential" - i.e., that has no voltage drop across it - but since there is no voltage drop across wires, this is exactly the same as our earlier criteria.)

As an example, let's consider the circuit we were analyzing, but return to Steps 1 and 2. As shown below, the junctions previously labeled as u_3 and ground are connected by a wire and are therefore a single node. We can label that entire node as ground. Similarly, the junctions previously labeled as u_1 and u_2 are also connected by a wire, so are also a single node. We can label that entire node as u_1.



Original procedure:
Labeling junctions



Simplified procedure:
Labeling nodes

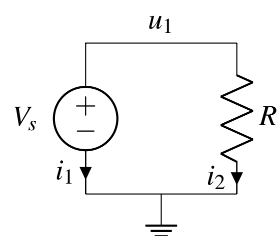
When we followed the original analysis procedure where we labeled *junctions*, we ended up with three unknown u's; by labeling only the *nodes*, we have simplified down to a single unknown u (u_1). In general, since wires are abundant in circuit diagrams, labeling only the nodes (instead of the junctions) will substantially reduce the number of variables.

11.2.2 Trivial Junctions

We define a **trival junction** to be a junction connecting only two elements. KCL dictates that the current entering the junction must be equal to the current exiting. Since there are only two elements, it follows that the two currents must be equal (as long as we label the direction of current flow to be the same – if not, the currents will simply be opposite in sign).

Therefore, another simplification to our analysis procedure is to **label the currents only in the non-wire elements** in our circuit. (Sometimes these currents are called *branch currents*). We can later find the current in any given wire by looking for a trivial junction between the wire and a non-wire element. When we use KCL, we can now consider *nodes* (instead of junctions) — i.e. the current flowing into the node is equal to the current leaving the node.

Returning to our example, if we repeat Step 3 (and assume labeled nodes rather than junctions, as explained in the previous section), we would now label only the current through the two non-wire elements: the voltage source and the resistor.



With this simplified approach, when we get to Step 5 (KCL), we would apply KCL at the node u_1, which would result in the equation:

$$-i_1 - i_2 = 0 \tag{1}$$

11.2.3 Summary of Simplified Procedure

By labeling nodes instead of junctions and labeling currents in non-wire elements only, we can greatly reduce the number of variables in our circuit analysis procedure, so this is what we will do in the future. Here's a summary of the steps:

- **Step 1:** Pick a **node** and label it as $u = 0$ ("ground"), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.
- **Step 2:** Label all remaining **nodes** as some "u_i", representing the voltage at each node relative to the ground node.
- **Step 3:** Label the current through every **non-wire** element in the circuit "i_n".
- **Step 4:** Add $+/-$ labels (indicating direction of voltage measurement) on each **non-wire** element by following the passive sign convention.

- **Step 5:** Use KCL to write equations at the labelled nodes.
- **Step 6:** Use Ohm's Law to write IV relationships of non-wire elements.
- **Step 7:** Solve system of equations using substitution.

11.2.4 Summary of Matrix-Oriented Procedure

- **Step 1 to 4:** Same as the simplified procedure above.
- **Goal** The goal is to set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the u_i's and i_n's defined in the previous steps.
- **Step 5:** If there are n **nodes** (including the ground node), use KCL on $(n-1)$ nodes to fill in $(n-1)$ rows of $\mathbf{A}$ and $\vec{b}$.
- **Step 6:** If there m **non-wire** elements, use the IV relationships of each non-wire element to fill in the remaining m equations (rows of $\mathbf{A}$ and values of $\vec{b}$).
- **Step 7:** Solve with your favorite technique from linear algebra!

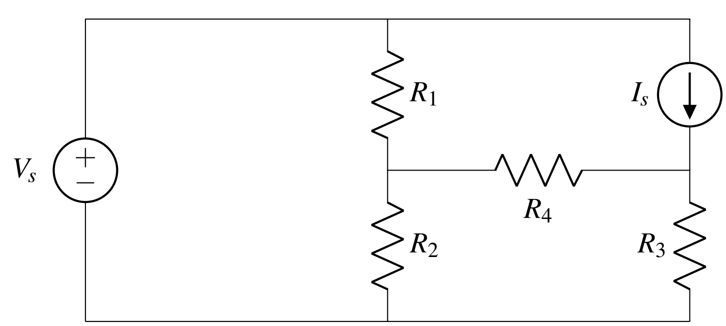
11.3 Example

11.3.1 Objective

Find all voltages (and currents) in an electronic circuit.

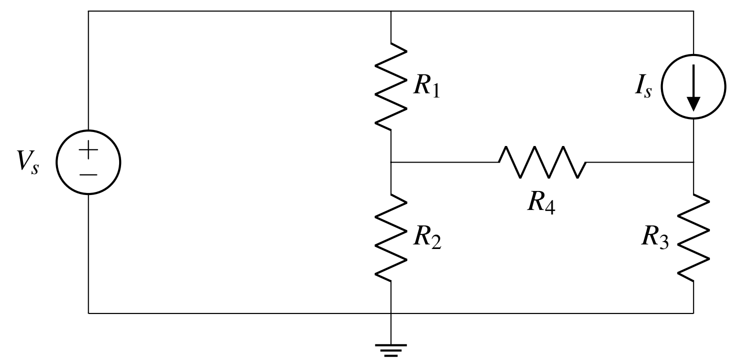
11.3.2 Procedure

The method proceeds steps described, illustrated below for the following example circuit:

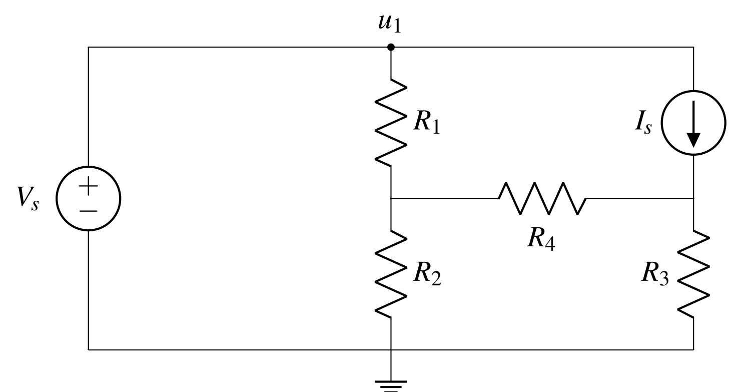


- **Step 1:** Reference Node

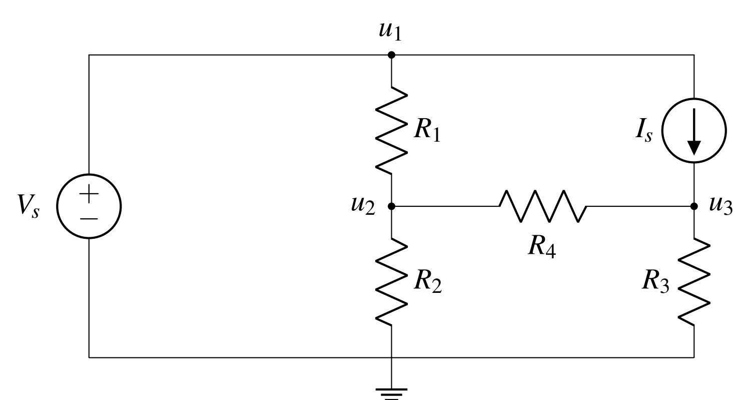
  Select a reference (ground) node. Any node can be chosen for this purpose. In this example, we choose the node at the bottom of the circuit diagram.



- **Step 2:** Label Nodes First lets look at the nodes with voltage set by Voltage Sources. Voltage sources set the voltage of the node they are connected to. In the example, there is only one source, V_s, and we label the corresponding source u_1 (names are arbitrary, but must be unique).

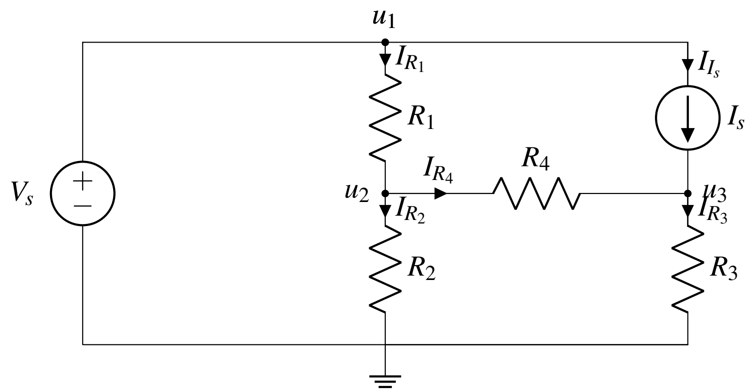


Now we label all remaining nodes in the circuit except the reference. In this example there are two, u_2 and u_3.



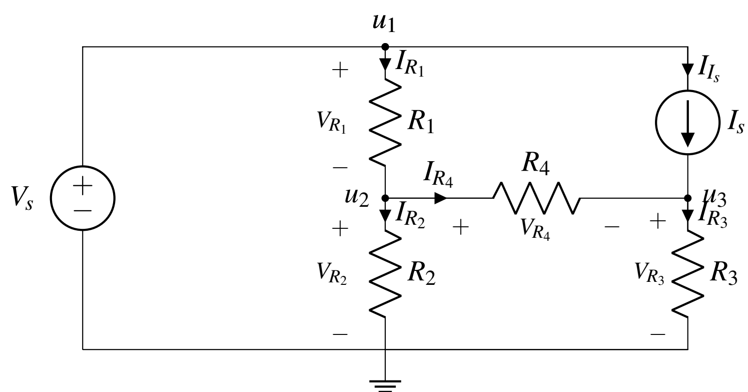
- **Step 3:** Label currents through non-wire elements

  The direction is arbitrary (top to bottom, bottom to top, it won't matter, but stick with your choice in subsequent steps). Then mark the element voltages following the passive sign convention discussed on the following page.



- **Step 4:** Label element potentials based on passive sign convention.

  The element voltage for I_s is not marked in the example since it will not be needed in the calculations below. Same for the voltage source. There is no harm in marking those, too.



- **Step 5:** KCL Equations

  Write KCL equations for all nodes with unknown voltage, u_2 and u_3 in the example. Refer to Note 11A for a reminder of how to write KCL equations.

  At u_2 we get (sum of all currents entering the node equals sum of currents exiting):

$$I_{R_1} = I_{R_2} + I_{R_4}$$

  Similar for u_3:

$$I_{R_4} + I_{I_s} = I_{R_3}$$

- **Step 6:** Element IV Relationships

  Find expressions for all element currents in terms of voltage and element characteristics (e.g. Ohm's law) for all circuit elements except voltage sources. In the example there are five, R_1, R_2, R_3, R_4, I_s.

Find expressions for element currents for all elements (except the voltage source) using their characteristics. Applying Ohm's law to the two resistors, we find that

$$
\begin{aligned}
I_{R_1} &= \frac{V_{R_1}}{R_1} \\
I_{R_2} &= \frac{V_{R_2}}{R_2} \\
I_{R_3} &= \frac{V_{R_3}}{R_3} \\
I_{R_4} &= \frac{V_{R_4}}{R_4} \\
I_{I_s} &= I_s
\end{aligned}
$$

We also have

$$u_1 = V_s.$$

- **Step 7:** Solve

  We can substitute all element voltages in your step 6 equations with node voltages. For example, $V_{R_1} = u_1 - u_2 = V_s - u_2$ and $V_{R_2} = u_2 - 0 = u_2$.

$$
\begin{aligned}
I_{R_1} &= \frac{V_s - u_2}{R_1} \\
I_{R_2} &= \frac{u_2}{R_2} \\
I_{R_3} &= \frac{u_3}{R_3} \\
I_{R_4} &= \frac{u_2 - u_3}{R_4} \\
I_{I_s} &= I_s
\end{aligned}
$$

  Now we substitute the expressions derived into the KCL equations from Step 5.

$$
\begin{aligned}
\frac{V_s - u_2}{R_1} &= \frac{u_2}{R_2} + \frac{u_2 - u_3}{R_4} \\
I_s + \frac{u_2 - u_3}{R_4} &= \frac{u_3}{R_3}
\end{aligned}
$$

  Let's make this a bit nicer by grouping the unknowns (u_2 and u_3) on the left side and the known terms on the right:

$$
\begin{aligned}
u_2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right) + u_3\left(-\frac{1}{R_4}\right) &= \frac{V_s}{R_1} \\
u_2\left(-\frac{1}{R_4}\right) + u_3\left(\frac{1}{R_3} + \frac{1}{R_4}\right) &= I_s
\end{aligned}
$$

Now we can solve for the unknown node voltages, u_2 and u_3 in the example. This is a good time to use linear algebra. First, rearrange terms to cast the equations into a matrix problem:

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_3}+\frac{1}{R_4} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{V_s}{R_1} \\ I_s \end{bmatrix}$$

Then compute the solution using Gaussian Elimination (or let the computer do the work, here using sympy):

```
from sympy import *
init_printing(use_unicode=True)

R1, R2, R3, R4 = symbols('R1 R2 R3 R4')
Y = Matrix([[ 1/R1+1/R2+1/R4, -1/R4], [-1/R4, 1/R3+1/R4]])

V1, I1 = symbols('V1 I1')
b = Matrix([ V1/R1, I1 ])

Vn1, Vn2 = linsolve((Y, b)).args[0]
```

Algebraic result:

```
>>> Vn1
    R2⋅(I1⋅R1⋅R3 + R3⋅V1 + R4⋅V1)
──────────────────────────────────────
R1⋅R2 + R1⋅R3 + R1⋅R4 + R2⋅R3 + R2⋅R4
>>> Vn2
  R3⋅R4⋅(I1⋅(R1⋅R2 + R1⋅R4 + R2⋅R4) + R2⋅V1)
───────────────────────────────────────────────
-R1⋅R2⋅R3 + (R3 + R4)⋅(R1⋅R2 + R1⋅R4 + R2⋅R4)
```

Numerical result:

```
>>> values = {R1:1, R2:2, R3:3, R4:4, I1:0.5, V1:1}
>>>
>>> f"Vn1 = {Vn1.evalf(3, subs=values)} V"
Vn1 = 0.739 V
>>> f"Vn2 = {Vn2.evalf(3, subs=values)} V"
Vn2 = 1.17 V
```

Branch Currents

Sometimes we want to solve for branch currents. These are easily obtained from the node voltages and element equations. For example, the current I_{R_4} through resistor R_4 is

$$I_{R_4} = \frac{V_{R_4}}{R_4} = \frac{u_2 - u_3}{R_4}$$

Numerical result using the values provided in Step 9 of 11.1.2: I_{R_4} = -0.109 A.

Reference

Reference: Schaum's Outline of Electric Circuits, Seventh Edition, Section 4.4.