EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 11

This homework is due November 12, 2021, at 23:59.
Self-grades are due November 15, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw11.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit each file to its respective assignment on Gradescope.

1. Reading Assignment

For this homework, please read Notes 18 and 19. They will provide an overview on operational amplifiers (op-amps), negative feedback, the "golden rules" of op-amps, and various op-amp configurations (non-inverting, inverting, buffers, etc). You are always encouraged to read beyond this as well.

(a) What are the two "golden rules" of ideal op-amps? When do these rules hold true?

Solution: The golden rules are the following:

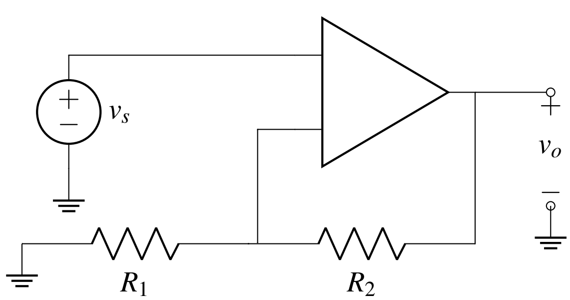
- The error signal going into the op-amp must be zero, i.e. $u_+ = u_-$. This rule only holds when there is negative feedback.
- The currents into the input terminals of the op-amp are zero, i.e. $I_+ = I_- = 0$. This rule holds regardless of whether there is negative feedback or not.

(b) What does the internal gain of an op-amp, A, mean? What is its value for an ideal op-amp? What about for a non-ideal one?

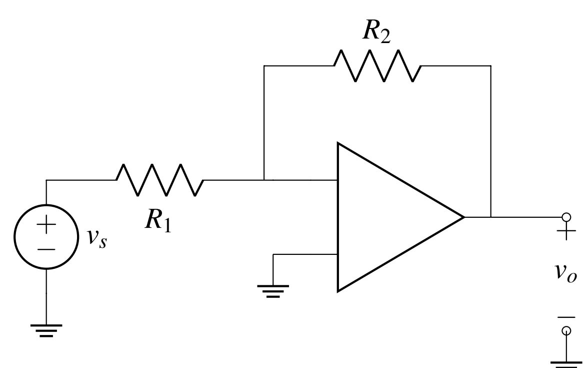
Solution: The internal gain of an op-amp, A is the ratio of the output voltage to the error voltage, i.e. A is given by $\frac{v_{out}}{u_+ - u_-}$. For ideal op-amps, $A \to \infty$. For non-ideal op-amps, A is finite.

2. Basic Amplifier Building Blocks

The following amplifier stages are used often in many circuits and are well known as (a) the non-inverting amplifier and (b) the inverting amplifier.



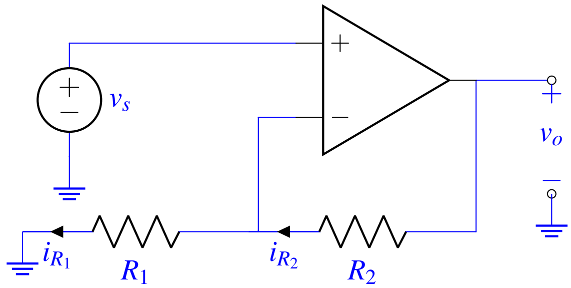
(a)



(b)

(a) Label the input terminals of the op-amp with $(+)$ and $(-)$ signs in Figure (a), so that it is in negative feedback. Then derive the voltage gain ($G = \frac{v_o}{v_s}$) of the non-inverting amplifier in Figure (a) using the Golden Rules. Why do you think this circuit is called a non-inverting amplifier?

Solution:



The $+$, $-$ should be labeled on the top and bottom of the op-amp, respectively. Now if we move the negative input of the op-amp u_- upward, $v_o = Av_{\text{error}} = A(u_+ - u_-)$ moves downward and as a result $u_- = \frac{R_1}{R_1+R_2}v_o$ moves downward. So the result of the initial stimulus goes in the opposite direction of the initial stimulus, which is the requirement for negative feedback.

By the Golden Rules, the voltage at the positive input terminal v_s must also set the voltage at the negative input terminal to be v_s. Our Golden Rules also tell us that no current can flow into the input terminals of the op-amp. Therefore, we can write a single KCL equation at the input node of the negative terminal as follows:

$$\begin{aligned} u_- &= u_+ = v_s \\ i_{R_1} &= i_{R_2} \\ \implies \frac{v_s}{R_1} &= \frac{v_o - v_s}{R_2} \end{aligned}$$

Rearranging and solving for v_o, we therefore obtain:

$$\begin{aligned} R_2 v_s &= R_1 v_o - R_1 v_s \\ \implies v_o &= \left(\frac{R_1 + R_2}{R_1}\right) v_s \end{aligned}$$

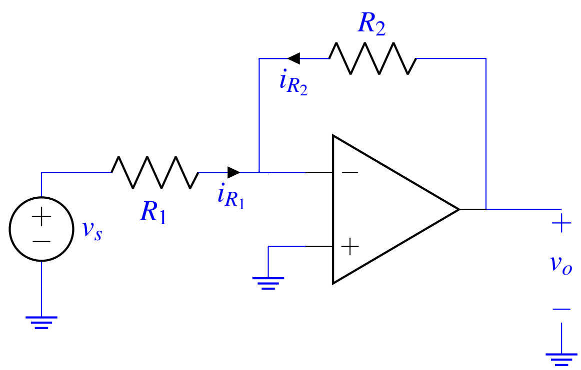
Note also that you may be familiar already with a faster way of solving this problem! Because no current flows into the negative input terminal of the op-amp, you may recognize R_1 and R_2 as simply forming a *voltage divider* over v_o. Therefore, the potential at the negative terminal is:

$$\begin{aligned} u_- = v_s &= v_o \left(\frac{R_1}{R_1 + R_2}\right) \\ \implies v_o &= \left(\frac{R_1 + R_2}{R_1}\right) v_s \\ \implies G = \frac{v_o}{v_s} &= \left(\frac{R_1 + R_2}{R_1}\right) \end{aligned}$$

This is called an *non-inverting amplifier* because the gain G is positive – it does not invert the input signal (in contrast to the amplifier in the next part of this problem).

(b) Label the input terminals of the op-amp with $(+)$ and $(-)$ signs in Figure (b), so that it is in negative feedback. Then derive the voltage gain ($G = \frac{v_o}{v_s}$) of the inverting amplifier using the Golden Rules. Can you explain why this circuit is called an inverting amplifier?

Solution:



The $+$, $-$ should be labeled on the bottom and top of the op-amp, respectively. Now if we move the negative input of the op-amp u_- upward, $v_o = Av_{\text{error}} = A(u_+ - u_-)$ moves downward and as a result u_- moves downward because of the following relationship:

$$\frac{v_o - u_-}{R_2} = \frac{u_- - v_s}{R_1}$$
$$\implies u_- = \frac{R_1}{R_1 + R_2} v_o + \frac{R_2}{R_1 + R_2} v_s$$

So the result of the initial stimulus goes in the opposite direction of the initial stimulus, which is the requirement for negative feedback.

Since the potential at the positive input terminal is $u_+ = 0$, the op-amp will act such that the potential at the negative input terminal is $u_- = 0$ as well (by the Golden Rules). Now, by KCL at the node with potential u_-:

$$i_{R_1} = i_{R_2}$$
$$\frac{v_s - 0}{R_1} + \frac{v_o - 0}{R_2} = 0$$

Solving this yields:

$$v_o = -\left(\frac{R_2}{R_1}\right) v_s$$

Thus, the voltage gain of this amplifier circuit is:

$$G = \frac{v_o}{v_s} = -\frac{R_2}{R_1}$$

This is called an *inverting amplifier* because the voltage gain G is *negative*, meaning it "inverts" its input signal.

(c) Using your toolkit of circuit topologies, design blocks that implement the following equations. Feel free to reference Discussion 10B for circuit topologies:

i. $v_o = 2v_s$

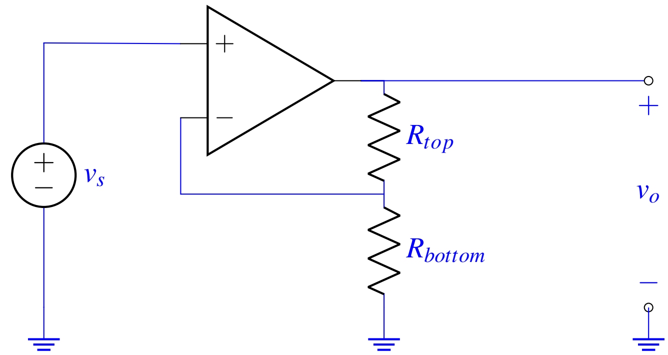
ii. $v_o = -3v_s + 8$

Solution:

i. We can use a non-inverting amplifier with gain 2.

$$\frac{v_o}{v_s} = \left(1 + \frac{R_{top}}{R_{bottom}}\right) = 2$$
$$\frac{R_{top}}{R_{bottom}} = 1$$

Any value for R_{top} and R_{bottom} is correct as long as they have the same resistance.



ii. We can use an inverting amplifier with reference voltage source. We need to determine values for R_f, R_s and V_{REF}.

$$v_o = v_s\left(-\frac{R_f}{R_s}\right) + V_{REF}\left(\frac{R_f}{R_s}+1\right) = -3v_s + 8$$

Matching the coefficients in this equation:

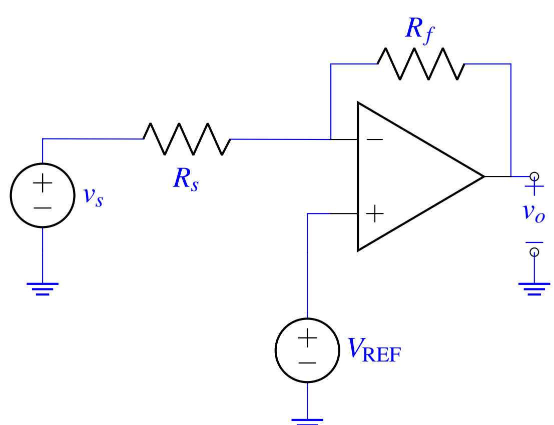
$$\frac{R_f}{R_s} = 3$$

$$V_{REF}\left(\frac{R_f}{R_s}+1\right) = 8$$

$$V_{REF}(3+1) = 8$$

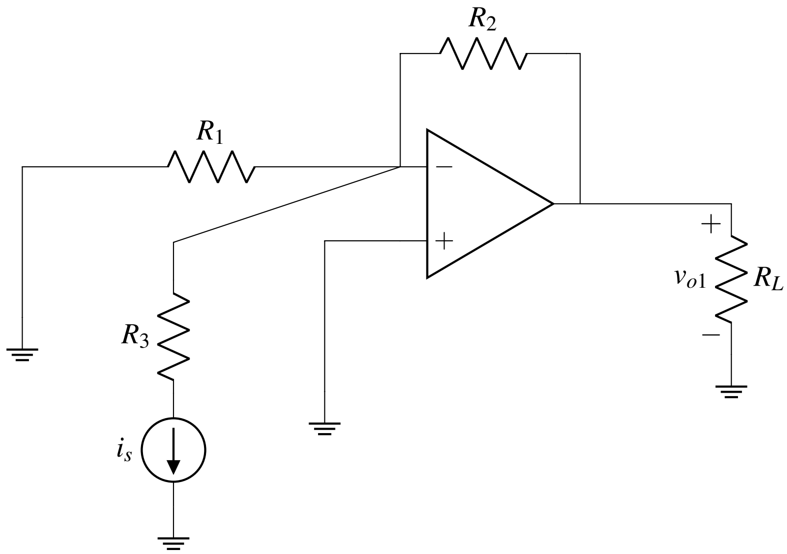
$$V_{REF} = 2V$$

Any values for R_f and R_s are correct as long as $\frac{R_f}{R_s} = 3$.



3. Amplifier with Multiple Inputs

(a) Use the Golden Rules to find v_{o1} for the circuit below.



Solution:

Applying the Golden Rules, we know that the positive and negative terminals must be at the same voltage. Thus, the voltage at the negative terminal of the op-amp is 0. The voltage drop across R_1 is 0 and no current flows through it. In addition, no current flows into the op-amp from the negative terminal due to its infinite input resistance (the negative terminal is connected to an "open" circuit).

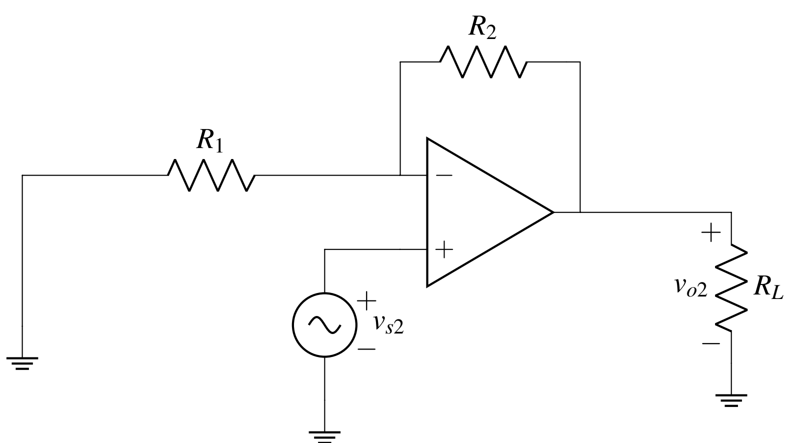
By KCL at the negative terminal of the op-amp, this means that the current going through R_3 and R_2 is i_s. Taking the positive terminal of R_2 to be on the right, the voltage drop across R_2 is v_{o1}. By Ohm's law, we conclude:

$$\frac{v_{o1}}{R_2} = i_s$$

Rearranging we get:

$$v_{o1} = i_s \cdot R_2$$

(b) Use the Golden Rules to find v_{o2} for the circuit below.



Solution:

Applying the Golden Rules, we know that the positive and negative terminals must be at the same voltage. Thus, the voltage at the negative terminal of the op-amp is $V^- = v_{s2}$. In addition, since no current can enter into the negative terminal of the op-amp, R_1 and R_2 are in series. This means that the voltage at the negative terminal of the op-amp can be expressed in terms of v_{o2} using the voltage divider formula:

$$v^- = v_{o2}\left(\frac{R_1}{R_1+R_2}\right)$$

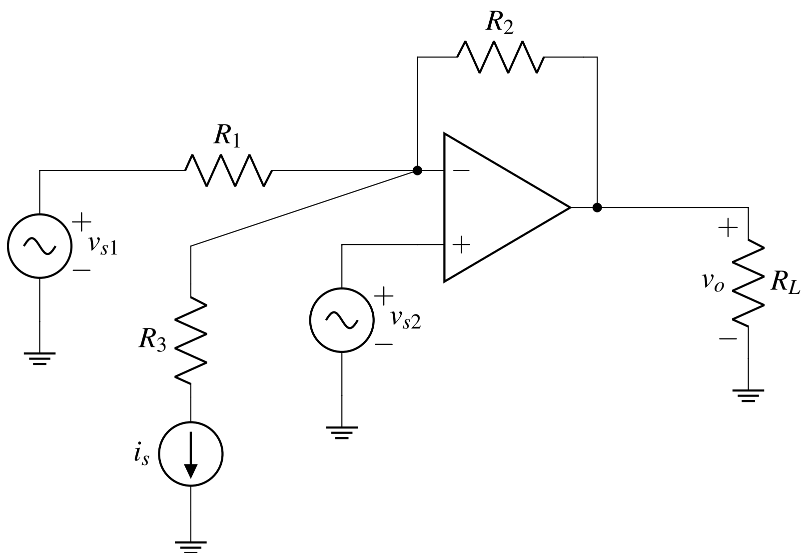
We also know that $v^- = v_{s2}$ and conclude:

$$v_{s2} = v_{o2}\left(\frac{R_1}{R_1+R_2}\right)$$

After rearranging, we have:

$$v_{o2} = v_{s2}\left(\frac{R_2}{R_1}+1\right)$$

(c) Use the Golden Rules to find the output voltage v_o for the circuit shown below.



Solution:

Applying the Golden Rules we know that the positive and negative terminals must be at the same voltage. Thus, the voltage at the negative terminal of the op-amp is $v^- = v_{s2}$. Then we write a KCL equation at the node connected to the minus terminal of the op-amp (recalling that no current flows into or out of the op-amp's terminals). All currents are defined as flowing out of the node:

$$i_{R_1} + i_{R_2} + i_{R_3} = 0$$

Because of the independent current source, we know:

$$i_{R_3} = i_s$$

By Ohm's law, we know:

$$i_{R_1} = \frac{v^- - v_{s1}}{R_1}$$

and

$$i_{R_2} = \frac{v^- - v_o}{R_2}$$

Then, substituting back into the original KCL equation, we have:

$$\frac{v^- - v_{s1}}{R_1} + \frac{v^- - v_o}{R_2} + i_s = 0$$

and substituting $v^- = v_{s2}$, we have:

$$\frac{v_{s2} - v_{s1}}{R_1} + \frac{v_{s2} - v_o}{R_2} + i_s = 0$$

which we rearrange to find v_o, giving:

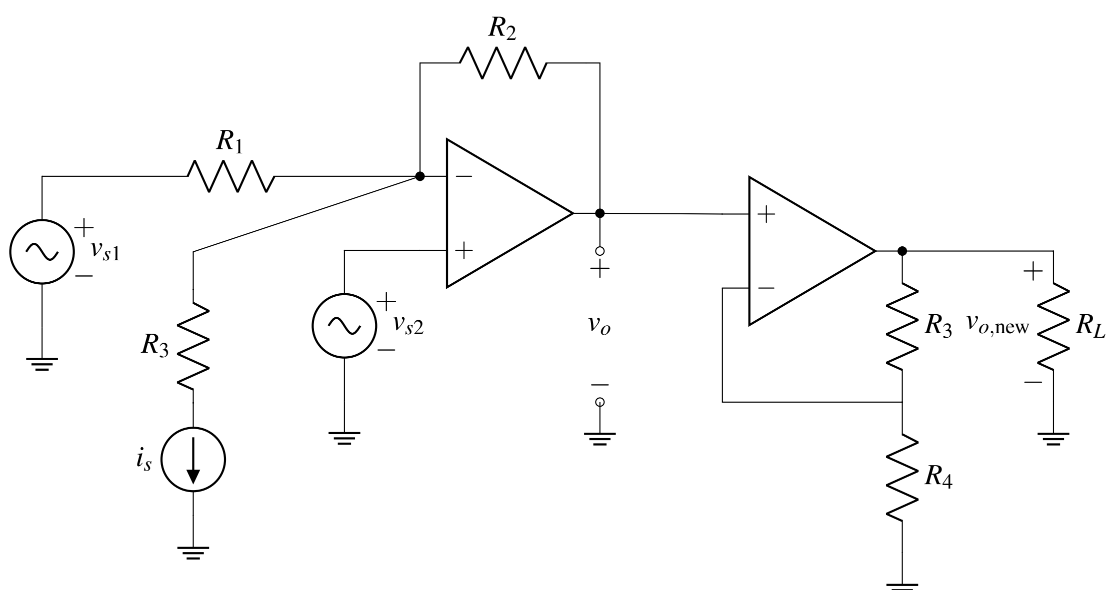
$$v_o = v_{s2}\left(1 + \frac{R_2}{R_1}\right) + i_s \cdot R_2 - \left(\frac{R_2}{R_1}\right) v_{s1}$$

(d) Use superposition and the answers to the first few parts of this problem to verify your answer to part c. *Hint: See if you can generate some combination of the circuits in a & b that is equivalent to the one in c.*

Solution: Using superposition we can analyze the circuit leaving only one source on at a time. If we leave on v_{s1} and turn off v_{s2} and i_s, then we have an inverting amplifier. If we leave on i_s and turn off v_{s1} and v_{s2}, then we have the circuit in (a). If we leave on v_{s2} and turn off v_{s1} and i_s, then we have the circuit in (b). From this we can see that v_o is the sum from the solutions.

$$v_o = -\frac{R_2}{R_1} v_{s1} + i_s R_2 + v_{s2} \frac{R_2 + R_1}{R_1}$$

(e) Now add a second stage as shown below. What is $v_{o,\text{new}}$? Does v_o change between part (c) and this part? Does the voltage $v_{o,\text{new}}$ depend on R_L?



Solution:

Adding the second stage does not change the voltages in the first stage. This is because the circuit connected to the positive and negative terminals of the first stage op-amp "sees" an open circuit/infinite input resistance in the op-amp.

Hence v_o remains unchanged from part (c).

$$v_o = -\left(\frac{R_2}{R_1}\right)v_{s1} + i_s \cdot R_2 + v_{s2}\left(\frac{R_2+R_1}{R_1}\right)$$

By the Golden Rules, the negative terminal of the second op-amp must have the same voltage as the plus terminal, which is v_o. No current can flow into the negative terminal, so R_3 and R_4 are in series and have the same current, so we know:

$$\frac{v_o}{R_4} = \frac{v_{o,new} - v_o}{R_3}$$

Therefore:

$$v_{o,new} = \left(\frac{R_3+R_4}{R_4}\right)v_o = \frac{R_3+R_4}{R_4}\left(-\frac{R_2}{R_1}\cdot v_{s1} + i_s \cdot R_2 + v_{s2}\cdot\frac{R_2+R_1}{R_1}\right)$$

Note that you could have directly used the non-inverting amplifier gain formula $(1+\frac{R_3}{R_4})$for this extra stage.

The output voltage **does not** depend on the load resistance R_L, since it is set by the dependent voltage source inside the op-amp. Remember that a voltage source will provide any amount of current necessary while maintaining its voltage constant. **That is the beauty of op-amps:** they provide isolation between stages because of the open circuit at the input and they get rid of the loading effect, since they can maintain the output voltage constant regardless of the load value.

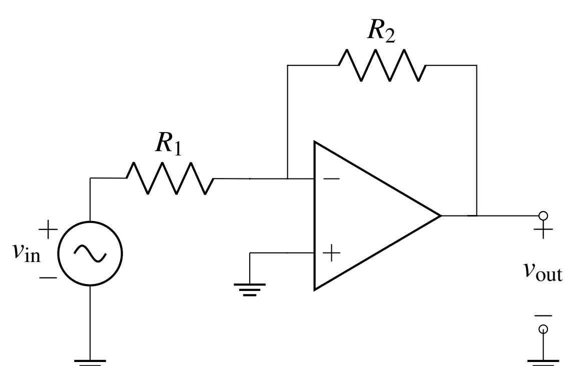
4. Cool For The Summer

You and a friend want to make a box that helps control an air conditioning unit based on both your inputs. You both have individual dials which you can use to control the voltage. An input of 0 V means that you want to leave the temperature as is. A **negative voltage input** means that you want to **reduce** the temperature. (It's hot out, so we will assume that you never want to increase the temperature – so no, we're not talking about a Berkeley summer...)

Your air conditioning unit, however, responds only to **positive voltages**. The higher the magnitude of the voltage, the stronger it runs. At zero, it is off. You also need a system that **sums up** both you and your friend's control inputs.

Therefore, you need a box that acts as an **an inverting summer** – *it outputs a weighted sum of two voltages where the weights are both negative*. The sum is weighted because one room is bigger, so you need to compensate for this.

(a) As a first step, derive v_{out} in terms of R_2, R_1, and v_{in}.
Hint: Have you solved for this particular amplifier configuration before? You can use your answer from the time you did this earlier.



Solution:

We will first need to check that the amplifier is in negative feedback in order to apply the Golden Rules. The amplifier is configured in negative feedback if, *after zeroing all independent sources*, when the negative input terminal voltage is elevated, the feedback moves it back downward. Going around the loop:

- We move the negative input voltage of the op amp upward
- The output voltage of the amplifier moves downward
- The negative input voltage moves downward with it

Thus, we've confirmed that the amplifier is in negative feedback.

Second, we perform KCL at the u_- terminal.

$$\frac{v_{\text{in}} - u_-}{R_1} + \frac{v_{\text{out}} - u_-}{R_2} = 0$$

Since we're in negative feedback, we can apply the Golden Rules.The voltages at the input terminals of the amplifier, u_- and u_+, respectively, must be held at the same voltage. In other words, $u_+ = u_- = 0\text{V}$.

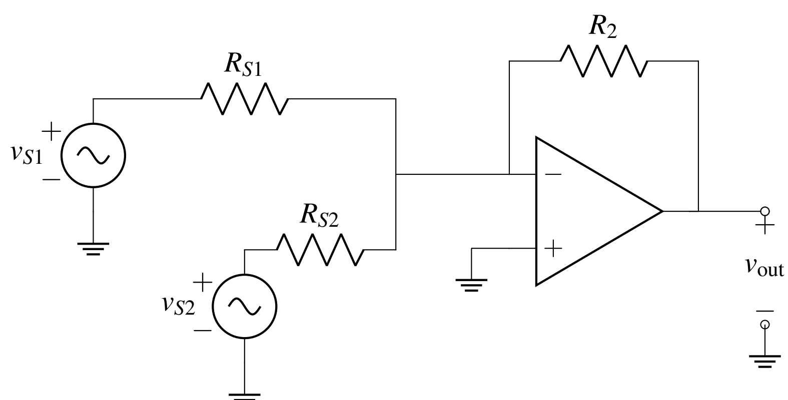
$$\frac{v_{\text{in}}}{R_1} + \frac{v_{\text{out}}}{R_2} = 0$$

$$v_{\text{out}} = v_{\text{in}} \left(-\frac{R_2}{R_1} \right)$$

The general inverting amplifier shown above has a voltage gain $G = \frac{v_{\text{out}}}{v_{\text{in}}} = -\frac{R_2}{R_1}$.

(b) Now we will add a second input to this circuit as shown below. Find v_{out} in terms of v_{S1}, v_{S2}, R_{S1}, R_{S2} and R_2.

Hint: You can solve this problem using either superposition or our tried-and-true KCL analysis.



Solution:

Method 1: Superposition

First, when considering v_{S1}, we zero out v_{S2}, and therefore we can disregard R_{S2}. The reason why we can disregard R_{S2} is because by the Golden Rules, we know that the voltage at the $-$ terminal of the op-amp must be equal to the voltage at the $+$ terminal. Therefore, both terminals of R_{S2} are at $0\,\text{V}$, and no current flows through R_{S2}. With this insight, we recognize that it becomes identical to the circuit in part (a), except with $V_{in} \to V_{S1}$ and $R_1 \to R_{S1}$.

Now apply the equation from part (a): $v_{\text{out}} = -\frac{R_2}{R_{S1}} v_{S1}$.

Similarly, when v_{S2} is on and v_{S1} is zeroed out, we disregard R_{S1} by the same argument leading to $v_{\text{out}} = -\frac{R_2}{R_{S2}} v_{S2}$.

Combining the two v_{out} equations from superposition, we get $v_{\text{out}} = -R_2\left(\frac{v_{S1}}{R_{S1}} + \frac{v_{S2}}{R_{S2}}\right)$.

Method 2: KCL without superposition

According to the Golden Rules, $u_- = u_+ = 0\text{V}$, so we can write a single KCL equation at the u_- node and solve:

$$\frac{v_{S1}}{R_{S1}} + \frac{v_{S2}}{R_{S2}} + \frac{v_{\text{out}}}{R_2} = 0$$

$$v_{\text{out}} = -v_{S1}\left(\frac{R_2}{R_{S1}}\right) - v_{S2}\left(\frac{R_2}{R_{S2}}\right)$$

(c) Let's suppose that you want $v_{\text{out}} = -\left(\frac{1}{4}v_{S1} + 2v_{S2}\right)$ where again v_{S1} and v_{S2} represent the input voltages from you and your friend's control knobs. Select resistor values such that the circuit from part (b) implements this desired relationship.

Solution: Using the configuration from the previous part, the conditions which need to be satisfied are:

- $\frac{R_2}{R_{S1}} = \frac{1}{4}$
- $\frac{R_2}{R_{S2}} = 2$

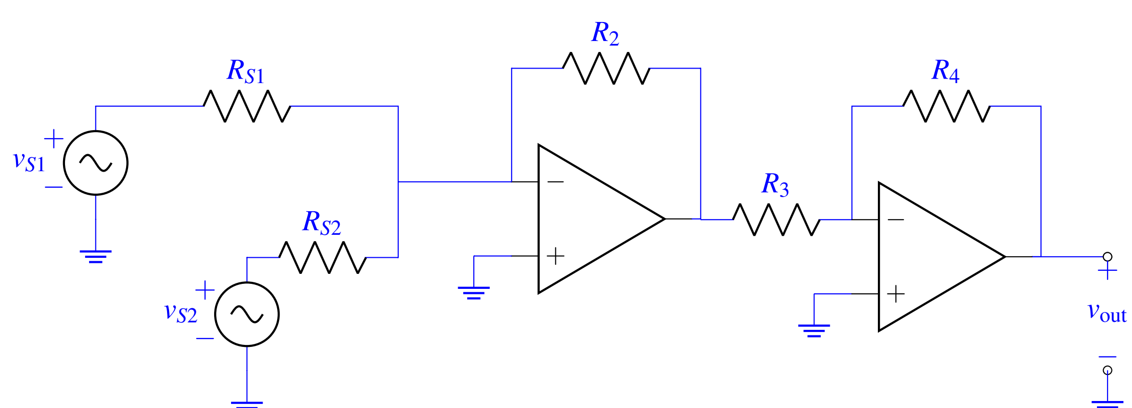
One possible set of values is $R_2 = 2\,\text{k}\Omega$, $R_{S1} = 8\,\text{k}\Omega$, and $R_{S2} = 1\,\text{k}\Omega$, but any combination of resistors which satisfies $R_{S1} = 4R_2 = 8R_{S2}$ are valid solutions.

(d) Suppose that you have a new AC unit that you want to use with your original control inputs v_{S1} and v_{S2}. **This unit, however, responds only to negative voltages – the opposite of your previous air conditioning unit, which only responded to positive input voltages.** The higher the magnitude of the negative voltage, the stronger the AC runs.

You want to modify your prior circuit for the new AC unit. Your circuit takes in two control voltages and outputs a weighted sum, but the sum should now become more negative as you increase your input voltages.

Hint: Consider adding another op-amp circuit to the output of your circuit from part (b), such that you invert the output of the op-amp circuit of part (b) ***without*** *adding additional gain.*

Solution:



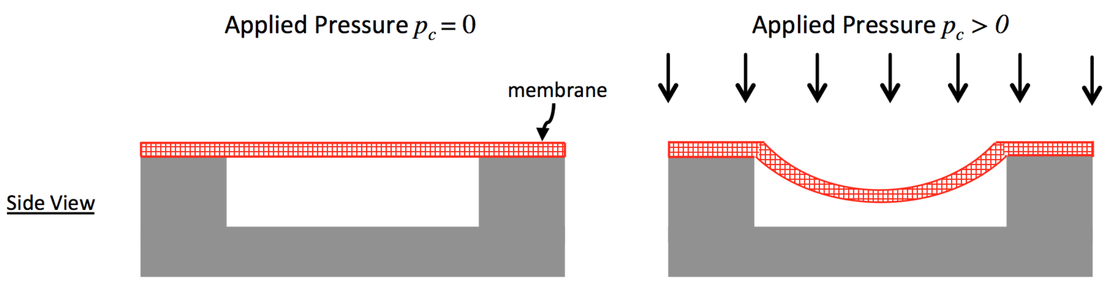
Here, we add another inverting op-amp stage with unity voltage gain, and we can pick any equal-valued resistors for R_3 and R_4.

5. Putting on the Pressure: Build Your Own InstantPot

Prof. Arias had a great experience with her automatic pressure cooker, so she was inspired to try and build her own. She's enlisting your help! The design of the pressure cooker uses a pressure sensor and a heating element. Whenever the pressure is below a set target value, an electronic circuit turns on the heating element.

Pressure Sensor Resistance

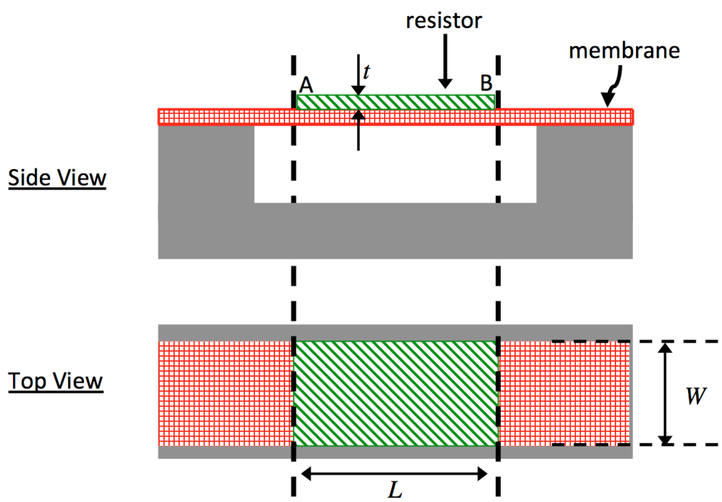
The first step is designing a pressure sensor. The figure below shows your design. As pressure p_c is applied, the flexible membrane stretches.



(a) You attach a resistor layer R_p with resistivity $\rho = 0.1\Omega\text{m}$, width W, length L, and thickness t to the pressure sensor membrane, as illustrated in the figure below. When the pressure $p_c = 0\,\text{Pa}$ (i.e. there is no applied pressure), $W = 1\,\text{mm}$, $L = L_0 = 1\,\text{cm}$, $t = 100\,\mu\text{m} = 100 \times 10^{-6}$ m.

R_{p0} **is the value of** R_p **when there is no applied pressure. Calculate** R_{p0}**.** Note that direction of current flow in the resistor is from A to B as marked in the diagram.

Solution: Resistance $R_{p0} = \frac{\rho \times L}{A} = \frac{\rho \times L_0}{W \times t} = \frac{0.1\Omega\text{m} \times 0.01\text{m}}{0.001\text{m} \times 100 \times 10^{-6}\text{m}} = 10\text{k}\Omega$.



(b) When pressure is applied, the length of the resistor L changes from L_0 and is a function of applied pressure p_c, and is given by

$$L = L_0 + \beta p_c,$$

where L_0 is the nominal length of the resistor with no pressure applied, and β is a constant with units m/Pa. As a result of the length change, the value of resistance R_p also changes from its nominal value R_{p0} (the value of R_p with no pressure applied).

Derive an expression for R_p as a function of resistivity ρ, width W, thickness t, nominal length L_0, constant β, and applied pressure p_c, when pressure is applied.

Note: The width and thickness of the resistor will also change with applied pressure. However, we ignore this to keep the math simple.

Solution: Resistance $R_p = \frac{\rho \times L}{A} = \frac{\rho \times L}{W \times t}$. Now when pressure is applied the length is given by

$$L = L_0 + \beta p_c$$

Plugging in the value of L_{p_c} we have:

$$R_p = \frac{\rho \times (L_0 + \beta p_c)}{W \times t}$$

(c) **Pressure Sensor Circuit Design**

For this sub-part and the following sub-parts, we will use a new model for pressure-sensitive resistance R_p. Assume that the resistance R_p is a function of applied pressure p_c according to the relationship $R_p = R_o \times \frac{p_c}{p_{\text{ref}}}$, where $R_o = 1\text{k}\Omega$, and $p_{\text{ref}} = 100\text{kPa}$.

To complete our sensor circuit, we would like to generate a voltage V_p that is a function of the pressure p_c.

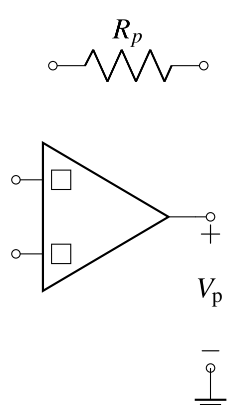
Complete the circuit below so that the output voltage V_p depends on the pressure p_c as:

$$V_p = -V_o \times \frac{p_c}{p_{\text{ref}}}, \text{ where } V_o = 1\,\text{V}.$$

Restrictions on your pressure sensor circuit design are as follows:

- **You may add at most one ideal voltage source and one additional resistor besides R_p to the circuit, but you must calculate their values and mark them in the diagram.**
- **Mark the positive and negative inputs of the operational amplifier with "+" and "-" symbols, respectively, in the boxes provided.**
- **Assume op-amp supply voltages V_{DD} and $V_{SS} = -V_{DD}$ are already provided.**

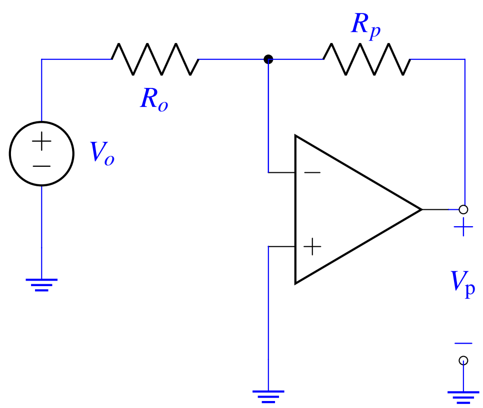
You may assume that the operational amplifier is ideal.



Solution: We can combine the relationships $R_p = R_o \times \frac{p_c}{p_{\text{ref}}}$ and $V_p = -V_o \times \frac{p_c}{p_{\text{ref}}}$ to get the relationship:

$$V_p = -V_o \times \frac{R_p}{R_o}$$

We can then use an inverting amplifier op-amp configuration to realize the circuit, where $R_o = 1\text{k}\Omega$ and $R_p = \frac{1\text{k}\Omega}{100\text{kPa}} \times p_c$.



(d) **Resistive Heating Element**

To heat the pressure cooker, you use a heating element with resistance R_{heat}. Calculate the value of R_{heat} such that the power dissipated is $P_{\text{heat}} = 1000\,\text{W}$ with $V_{\text{heat}} = 100\,\text{V}$ applied across the heating element.

Solution: $P_{heat} = V_{heat} I_{heat} = V_{heat} \times \frac{V_{heat}}{R_{heat}}$. Therefore, $R_{heat} = \frac{V_{heat}{}^2}{P_{heat}} = \frac{(100V)^2}{1000\text{W}} = 10\Omega$.

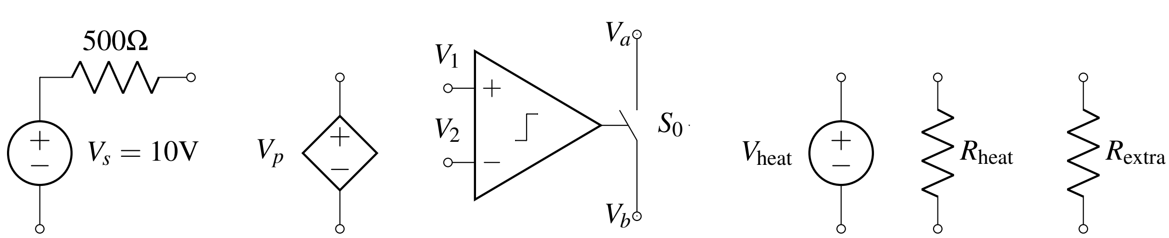
(e) **Pressure Regulation**

You are finally ready to complete the design of your pressure cooker.

Using all of the circuit elements below, make a circuit that will turn the heater on (i.e. will cause a current to flow through R_{heat}) when the pressure is less than 500 kPa, and off (i.e. will cause no current to flow through R_{heat}) when the pressure is greater than 500 kPa.

The elements are:

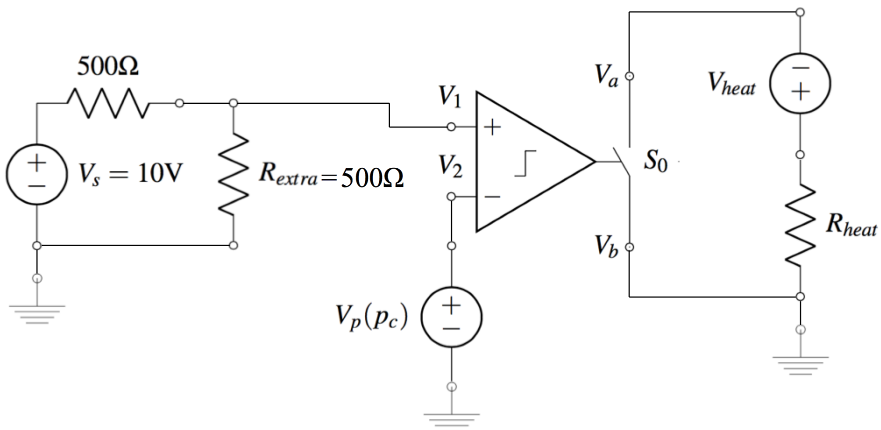
- A voltage source $V_s = 10\text{V}$ in series with a resistance of 500Ω.
- A dependent voltage source $V_p = V_o \times \frac{p_c}{p_{\text{ref}}}$, with $V_o = 1\text{V}$ and $p_{\text{ref}} = 100\text{kPa}$. (This is a voltage source whose voltage is a function of pressure p_c, unrelated to any previous parts of the question.)
- A comparator that controls switch S_0. The switch is normally opened (i.e. an open circuit between nodes V_a and V_b), and is closed only when $V_1 > V_2$ (i.e. a short circuit between nodes V_a and V_b).
- The heater supply ($V_{\text{heat}} = 100\text{V}$).
- The heater resistor R_{heat}.
- One additional resistor R_{extra} that can have **any value**.
- Assume comparator supply voltages V_{DD} and $V_{SS} = -V_{DD}$ are already provided.



Solution: We can compute $V_p = 5V$ for $p_c = 500$kPa. $5V$ is the voltage that we want to compare against the output of V_p in order to make sure the heater is on when the pressure p_c is less than 500kPa and off when the pressure p_c is greater than 500kPa.

We were able to generate a 5V reference voltage by using R_{extra} to make a voltage divider with the voltage source V_s. By setting R_{extra} equal to the voltage source's associated resistance, the output of the voltage divider is 5V. We connect this to the positive input of the comparator. We then connect the source V_p to the negative input of the comparator.

Finally, we connect the voltage source V_{heat} in series with the resistor R_{heat} so that when S_0 is closed (i.e. when the pressure p_c is less than 500kPa) power will be delivered to R_{heat}. Note that the polarity of V_{heat} doesn't matter as the resistor will always dissipate energy.



6. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.