EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 5

This homework is due Friday, February 25, 2022 at 23:59.
Self-grades are due Monday, February 28, 2022 at 23:59.

Submission Format
Your homework submission should consist of **one** file.

1. **Reading Assignment**

For this homework, please read Note 7, 8, and 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. You are always welcome and encouraged to read beyond this as well.

2. **Subspaces, Bases and Dimension**

For each of the sets $\mathbb{U}$ (which are subsets of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

(a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: We test the three properties of a subspace:

i. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{v}_2 = \alpha\vec{v}_1$, where α is a scalar. Here

$$\vec{v}_2 = \alpha\vec{v}_1 = \alpha \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2(\alpha x_1+\alpha y_1) \\ \alpha x_1 \\ \alpha y_1 \end{bmatrix} = \begin{bmatrix} 2(x_i+y_i) \\ x_i \\ y_i \end{bmatrix},$$

where $x_i = \alpha x_1$ and $y_i = \alpha y_1$. Hence, $\vec{v}_2 = \alpha\vec{v}_1$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} 2(x_1+y_1) \\ x_1 \\ y_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2(x_2+y_2) \\ x_2 \\ y_2 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 2(x_1+y_1)+2(x_2+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_1+x_2+y_1+y_2) \\ x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} 2(x_3+y_3) \\ x_3 \\ y_3 \end{bmatrix},$$

where $x_3 = x_1+x_2$ and $y_3 = y_1+y_2$ Hence, $\vec{v}_3$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v_0} = \begin{bmatrix} 2(x_0+y_0) \\ x_0 \\ y_0 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector $\vec{v_0} = \begin{bmatrix} 2(0+0) \\ 0 \\ 0 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

where x and y are free variables. So $\mathbb{U}$ can be expressed as span $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence the basis is given by the set: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Dimension = 2.

(b) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

Solution:

Again we check the three properties of a subspace:

i. Now let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{v_2} = \alpha \vec{v_1}$, where α is a scalar. Here

$$\vec{v_2} = \alpha \vec{v_1} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 + \alpha \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ (\alpha z_1 + \alpha - 1) + 1 \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i + 1 \end{bmatrix},$$

where $x_i = \alpha x_1$, $y_i = \alpha y_1$ and $z_i = \alpha z_1 + \alpha - 1$. Hence, $\vec{v_2} = \alpha \vec{v_1}$ is a member of the set as well and the set is closed under scalar multiplication.

ii. Let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1+1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2+1 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v_3} = \vec{v_1} + \vec{v_2}$:

$$\vec{v_3} = \vec{v_1} + \vec{v_2} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2+2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ (z_1+z_2+1)+1 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \\ z_3+1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$ and $z_3 = z_1 + z_2 + 1$. Hence, $\vec{v_3}$ is a member of the set as well and the set is closed under vector addition.

iii. Let $\vec{v_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0+1 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$, $y_0 = 0$ and $z_0 = -1$. So the vector $\vec{v_0} = \begin{bmatrix} 0 \\ 0 \\ -1+1 \end{bmatrix} = \vec{0}$. So the zero vector is contained in this set.

Hence we can decide that $\mathbb{U}$ is a subspace of $\mathbb{R}^3$. Any vector in the subspace can be written as:

$$\begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (z+1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z_{new} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where x, y and $z_{new} = z+1$ are free variables. So $\mathbb{U}$ can be expressed as span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Hence the basis is given by the set: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The dimension is 3, which makes $\mathbb{U}$ the same as $\mathbb{R}^3$.

(c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+1 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{v}_2 = \alpha \vec{v}_1$, where α is a scalar. Here

$$\vec{v}_2 = \alpha \vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha \end{bmatrix} \neq \begin{bmatrix} x_i \\ y_i \\ x_i + 1 \end{bmatrix},$$

where $x_i = \alpha x_1$ and $y_i = \alpha y_1$. Hence, $\vec{v}_2 = \alpha \vec{v}_1$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1+1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ x_2+1 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 + 2 \end{bmatrix} \neq \begin{bmatrix} x_3 \\ y_3 \\ x_3 + 1 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, $\vec{v}_3$ is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ x_0+1 \end{bmatrix}$ be a member of the set. The first and third elements cannot both be zero regardless of the value chosen for x_0. So the zero vector is not contained in this set.

Hence we can decide that $\mathbb{U}$ is not a subspace of $\mathbb{R}^3$. **Note that for full credit you only have to show that one of the properties is violated, you don't have to show all three.**

(d) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+y^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

Solution: Again we check the three properties of a subspace:

i. Now let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1^2 \end{bmatrix}$ be a member of the set $\mathbb{U}$. Assume $\vec{v}_2 = \alpha\vec{v}_1$, where α is a scalar. Here

$$\vec{v}_2 = \alpha\vec{v}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + \alpha y_1^2 \end{bmatrix} \neq \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha x_1 + (\alpha y_1)^2 \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ x_i + y_i^2 \end{bmatrix},$$

where $x_i = \alpha x_1$ and $y_i = \alpha y_1$. Hence, $\vec{v}_2 = \alpha\vec{v}_1$ is not a member of the set and the set is not closed under scalar multiplication.

ii. Let $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1^2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ x_2 + y_2^2 \end{bmatrix}$ be members of the set $\mathbb{U}$. Now, let us assume $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$:

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + x_2 + y_1^2 + y_2^2 \end{bmatrix} \neq \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) + (y_1 + y_2)^2 \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \\ x_3 + y_3^2 \end{bmatrix},$$

where $x_3 = x_1 + x_2$, and $y_3 = y_1 + y_2$. Hence, $\vec{v}_3$ is not a member of the set and the set is not closed under vector addition.

iii. Let $\vec{v}_0 = \begin{bmatrix} x_0 \\ y_0 \\ x_0 + y_0^2 \end{bmatrix}$ be a member of the set, where we choose $x_0 = 0$ and $y_0 = 0$. So the vector $\vec{v}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 + 0^2 \end{bmatrix} = \vec{0}$ is contained in this set.

Since two of the three properties do not hold, we can decide that $\mathbb{U}$ is not a subspace of $\mathbb{R}^3$.

Just showing that one of the three properties is violated is enough to prove that a subset is not a subspace. However, in order to prove that a subset is a subspace, you have to show that all three properties hold.

3. Finding Null Spaces and Column Spaces

Learning Objectives: *Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as Null($\mathbf{A}$) and the definition can be written in set notation as:

$$Null(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of $\mathbf{A}$'s columns. The column space can be notated as Col($\mathbf{A}$) or range($\mathbf{A}$) and the definition can be written in set notation as:

$$Col(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors - i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $\text{Col}(\mathbf{A})$?

Solution: If you are stuck solving a problem like this, consider concrete examples. We want to find the maximum possible number of linearly independent column vectors, so we look for examples and check if we can exceed certain values.

Consider the following example matrix, where the entries marked with $*$ are arbitrary values:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

Here all 5 columns are $\in \mathbb{R}^3$. The first three columns are linearly independent, so at least three linearly independent columns are achievable. The first three columns span $\mathbb{R}^3$, therefore any choice of fourth and fifth columns, also in $\mathbb{R}^3$, can be written as a linear combination of the first three columns. This means that we cannot exceed three linearly independent columns. Thus the maximum number of linearly independent column vectors is 3. In general, if $m < n$, then the columns of $\mathbf{A} \in \mathbb{R}^{m\times n}$ will always be linearly dependent, since you cannot have more than m linearly independent columns in $\mathbb{R}^m$.

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\text{Col}(\mathbf{A})$ (i.e. a basis for $\text{Col}(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $\text{Col}(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

Solution: $\text{Col}(\mathbf{A})$ is the space spanned by its columns, so the set of all columns is a valid span for $\text{Col}(\mathbf{A})$. However, we are asking you to choose a subset of the columns and still span $\text{Col}(\mathbf{A})$, as we showed in part (a). To find the minimum number of columns needed and determine the dimension of $\text{Col}(\mathbf{A})$, we can remove vectors from the set of columns until we are left with a linearly independent set.

By inspection, the second, fourth, and fifth columns can be omitted from a set of columns as they can be expressed as linear combinations of the first and third columns. Thus the dimension of $\mathbf{A}$ is 2.

One set spanning $\text{Col}(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Another valid set of vectors which span $\text{Col}(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Note with this second set, none of the columns of $\mathbf{A}$ appear. Despite this, the span of this set will still be equal to $\text{Col}(\mathbf{A})$, which for this matrix is the set of all vectors in $\mathbb{R}^3$ with zero third entry. Geometrically, both of these solutions span the same plane, i.e. the xy-plane in the 3D space.

Give yourself full credit if you recognized that the dimension was 2, and if you had a *minimum* set of vectors that spans $\text{Col}(\mathbf{A})$.

(c) Find a *minimum* set of vectors that span $\text{Null}(\mathbf{A})$ (i.e. a basis for $\text{Null}(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $\text{Null}(\mathbf{A})$?

Solution:

Finding $\text{Null}(\mathbf{A})$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We observe that x_2, x_4, and x_5 are free variables, since they correspond to the columns with no pivots. Thus, we let $x_2 = a$, $x_4 = b$, and $x_5 = c$. Now we rewrite the equations as:

$$\begin{aligned} x_1 &= -a + 2b - 3c \\ x_2 &= a \\ x_3 &= b - c \\ x_4 &= b \\ x_5 &= c \end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\text{Null}(\mathbf{A})$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of $\text{Null}(\mathbf{A})$ is 3, as it is the minimum number of vectors we need to span it.

(d) Find the sum of the dimensions of $\text{Null}(\mathbf{A})$ and $\text{Col}(\mathbf{A})$. What do you notice about this sum in relation to the dimensions of $\mathbf{A}$?

Solution: The dimensions of $\text{Col}(\mathbf{A})$ and $\text{Null}(\mathbf{A})$ add up to the number of columns in $\mathbf{A}$. This is true of all matrices. This relates to what is known as the rank-nullity theorem; however we will not be covering this in 16A. You'll get to explore this in 16B.

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span Col($\mathbf{B}$) (i.e. a basis for Col($\mathbf{B}$)). What is the minimum number of vectors required to span the Col($\mathbf{B}$)?

Solution:

We see that the first two column vectors of $\mathbf{B}$ are linearly independent and sufficient to span Col($\mathbf{B}$), since the third column is trivial (all zeros) and does not contribute anything to the span. Therefore, Col($\mathbf{B}$) has dimension 2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(f) You are given the following matrix $\mathbf{G}$. Find a *minimum* set of vectors that span Null($\mathbf{G}$), i.e. a basis for Null($\mathbf{G}$).

$$\mathbf{G} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

Solution: To find Null($\mathbf{G}$), we wish to solve for all $\vec{x}$ such that $\mathbf{G}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 2 & -4 & 4 & 8 & 0 \\ 1 & -2 & 3 & 6 & 0 \\ 2 & -4 & 5 & 10 & 0 \\ 3 & -6 & 7 & 14 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 1 & -2 & 3 & 6 & 0 \\ 2 & -4 & 5 & 10 & 0 \\ 3 & -6 & 7 & 14 & 0 \end{array}\right] \qquad R_1 \leftarrow \frac{1}{2}R_1$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 - 3R_1 \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_1 \leftarrow R_1 - 2R_2 \\ R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2 \end{array}$$

Vectors in Null($\mathbf{G}$) satisfy the following equations:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} \implies \begin{aligned} x_1 - 2x_2 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

We then assign free variables $x_2 = a$ and $x_4 = b$ and substitute in:

$$\begin{aligned} x_1 &= 2a \\ x_2 &= a \\ x_3 &= -2b \\ x_4 &= b \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, Null(G) is spanned by the vectors:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(g) For the following matrix $\mathbf{D}$, find Col($\mathbf{D}$) and its dimension, and Null($\mathbf{D}$) and its dimension.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find Col($\mathbf{D}$), we identify the linearly independent columns of $\mathbf{D}$ by inspection. The second column is a scaled version of the first column. The third column is linearly independent from the first and second columns, since it is not a scaled version of the first column. Finally, the fourth column is simply the first column minus the third column and thus is linearly dependent with respect to prior columns.

So we conclude that the linearly independent columns of $\mathbf{D}$ are the first and third columns so that a basis for Col($\mathbf{D}$) is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \end{bmatrix} \right\}$$

and thus the dimension of Col($\mathbf{D}$) is 2.

To find Null($\mathbf{D}$), we can row reduce the matrix to find solutions to $\mathbf{D}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & -1 & -3 & 4 & 0 \\ 3 & -3 & -5 & 8 & 0 \\ 1 & -1 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since we only have pivots in the first and third columns, we can assign the free variables $x_2 = s$ and $x_4 = t$. We can write all solutions to $\mathbf{D}\vec{x} = \vec{0}$ as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

A basis for Null($\mathbf{D}$) is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and thus the dimension of Null($\mathbf{D}$) is 2.

4. Linear Dependence in a Square Matrix

Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent. You can use the following conclusion in your proof:

If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

Solution:

Let $\vec{a}_1, \vec{a}_2, \dots \vec{a}_n$ be the columns of A. By the definition of linear dependence, there exist scalars, $c_1, c_2, \dots c_n$, not all zero, such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = \vec{0} \tag{1}$$

We define $\vec{c}$ to be a vector containing the c_i's as follows: $\vec{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^T$, where $\vec{c} \neq \vec{0}$ by the definition of linear dependence. We can write Eq. 1 in matrix vector form:

$$A\vec{c} = \vec{0} \tag{2}$$

Let's use the first hint: How many solutions are there to the equation $A\vec{x} = \vec{0}$? We know from Eq. 2 that $\vec{c}$ is a solution, but we can also show that $\alpha\vec{c}$ is a solution for any α:

$$A(\alpha\vec{c}) = \alpha\vec{0} = \vec{0} \tag{3}$$

Since $\vec{c}$ is not zero, every multiple of $\vec{c}$ is a different solution. Therefore there are infinite solutions to the equation $A\vec{x} = \vec{0}$.

What can we say about the result of Gaussian elimination if there are infinite solutions? We know that if there are infinite solutions, there must be a free variable after Gaussian elimination. In other words, there must be a column in the row reduced matrix with no leading entry. Therefore, there must be fewer leading entries than the number of columns. Since the matrix A is square, it has the same number of rows as columns, so there must be fewer leading entries than the number of rows. That means there is at least one row with no leading entry, which is equivalent to saying there must be one row that's all zeros in the row reduced matrix.

Finally, we were given that if there is a row of all zeros in the row reduced matrix, then the rows of A must be linearly dependent.

5. Mechanical Determinants

For each of the following matrices, compute their determinant and state whether they are invertible.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. **Solution:**

We can use the form of a 2×2 determinant from lecture:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Therefore,

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 = 6$$

Since the determinant is not 0, the matrix is invertible.

(b) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Solution:

$$\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 1 \cdot 0 = 6$$

Since the determinant is not 0, the matrix is invertible.

(c) $\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$.

Solution:

$$\det\left(\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}\right) = 6 \cdot 6 - 9 \cdot 4 = 0$$

Since the determinant is 0, the matrix is non-invertible. Note that the columns of the matrix are linearly dependent.

(d) $\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

Solution: To find the determinant of a 3 by 3 matrix, we can use the formula:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & d \\ g & h \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}\right) = -4 \cdot \det\left(\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}\right)$$

$$\begin{aligned} &= -4 \cdot [(1 \cdot 1) - (-3 \cdot 3)] - 2 \cdot [(5 \cdot 1) - (-3 \cdot 7)] + 1 \cdot [(5 \cdot 3) - (1 \cdot 7)] \\ &= -4 \cdot [10] - 2 \cdot [26] + 1 \cdot [8] \\ &= -84 \end{aligned}$$

Since the determinant is not 0, the matrix is invertible.

(e) $\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

Solution:

$$\det\left(\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}\right) = -4 \cdot \det\left(\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\right) - 0 \cdot \det\left(\begin{bmatrix} 5 & -3 \\ 7 & 1 \end{bmatrix}\right) + 0 \cdot \det\left(\begin{bmatrix} 5 & 1 \\ 7 & 3 \end{bmatrix}\right)$$

$$= -4 \cdot [(1 \cdot 1) - (-3 \cdot 3)] - 0 + 0$$

$$= -40$$

Since the determinant is not 0, the matrix is invertible.

(f) $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Solution:

This is a diagonal matrix, so we know that determinant is the product of the diagonal entries, 4216.

We can also see this by using Gaussian elimination on the matrix to reduce it to the identity and knowing that scaling a row by a constant also scales the determinant by that constant. Keeping track of all changes made in our Gaussian elimination:

i. Multiply R_1 by $-\frac{1}{4}$. Denote $c_1 = -\frac{1}{4}$ for bookkeeping.

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

ii. Multiply R_2 by $\frac{1}{17}$. Denote $c_2 = \frac{1}{17}$ for bookkeeping.

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

iii. Multiply R_3 by $-\frac{1}{31}$. Denote $c_3 = -\frac{1}{31}$ for bookkeeping.

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

iv. Multiply R_4 by $\frac{1}{2}$. Denote $c_4 = \frac{1}{2}$ for bookkeeping.

$$\mathbf{A}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

v. The determinant of this last matrix is known to be 1. Now we trace our steps, knowing that scaling a row by a constant c_i scales the determinant by c_i. Suppose the initial matrix is $\mathbf{A}$. Then the following must be true:

$$\det(\mathbf{I}) = c_4 \cdot \det(\mathbf{A}_4) = c_4 \cdot c_3 \cdot \det(\mathbf{A}_3) = c_4 \cdot c_3 \cdot c_2 \cdot \det(\mathbf{A}_2) = c_4 \cdot c_3 \cdot c_2 \cdot c_1 \cdot \det(\mathbf{A})$$

We know $\det(\mathbf{I}) = 1$, so we substitute and solve for $\det(\mathbf{A})$:

$$\det(\mathbf{I}) = c_4 \cdot c_3 \cdot c_2 \cdot c_1 \cdot \det(\mathbf{A}) = \frac{1}{2} \cdot \frac{-1}{31} \cdot \frac{1}{17} \cdot \frac{-1}{4} \cdot \det(\mathbf{A}) = \frac{1}{4216} \det(\mathbf{A})$$

$$\det(\mathbf{A}) = 4216 \cdot \det(\mathbf{I}) = 4216 \cdot 1 = 4216,$$

so the final answer is 4216. Notice that this is exactly the product of the diagonal entries, which is what we saw from the definition must be true for matrices that have zeros everywhere except on the main diagonal.
Since the determinant is not 0, the matrix is invertible.

6. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

Self-grading note: For this subproblem and the following subproblems which involve computing eigenvectors, give yourself full credit if the eigenvector(s) you calculated is/are a scaled (i.e, multiplied by a real valued α) version of the eigenvector(s) given in the solutions.

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input. And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\det(\begin{bmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}) = 0$$

$$(5-\lambda)(2-\lambda) - 0 = 0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

where x is a free variable.

Any vector in span$\{\begin{bmatrix}1\\0\end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 5$.

$\lambda = 2$:

$$\begin{bmatrix}3 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies x = 0 \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\y\end{bmatrix}$$

where y is a free variable.

Any vector in span$\{\begin{bmatrix}0\\1\end{bmatrix}\}$ is an eigenvector of the matrix with corresponding eigenvalue $\lambda = 2$.

(b) $\mathbf{A} = \begin{bmatrix}22 & 6\\6 & 13\end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$\det(\begin{bmatrix}22-\lambda & 6\\6 & 13-\lambda\end{bmatrix}) = 0$$

$$\begin{aligned}(22-\lambda)(13-\lambda) - 36 &= 0\\250 - 35\lambda + \lambda^2 &= 0\\(\lambda-10)(\lambda-25) &= 0\\\implies \lambda &= 10, 25\end{aligned}$$

$\lambda = 10$:

$$\mathbf{A}\vec{x} = 10\vec{x} \implies (\mathbf{A} - 10\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}22 & 6\\6 & 13\end{bmatrix} - \begin{bmatrix}10 & 0\\0 & 10\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}12 & 6\\6 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 2x + y = 0 \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-2x\end{bmatrix}$$

where x is a free variable.

Any vector that lies in span$\{\begin{bmatrix}1\\-2\end{bmatrix}\}$ is an eigenvector with corresponding eigenvalue $\lambda = 10$.

$\lambda = 25$:

$$\mathbf{A}\vec{x} = 25\vec{x} \implies (\mathbf{A} - 25\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix}22 & 6\\6 & 13\end{bmatrix} - \begin{bmatrix}25 & 0\\0 & 25\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies \begin{bmatrix}-3 & 6\\6 & -12\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies \begin{bmatrix}-1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \implies 2y = x \implies \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2y\\y\end{bmatrix}$$

where y is a free variable.

Any vector that lies in span$\{\begin{bmatrix}2\\1\end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 25$.

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Solution:

This can also be seen by inspection. The matrix is not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, which belongs in the **nullspace of the matrix**.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

Alternatively, we can explicitly calculate.

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$
$$\lambda^2 - 5\lambda = 0 \implies \lambda(\lambda-5) = 0$$
$$\lambda = 0, 5$$

$\lambda = 0$:

$$\mathbf{A}\vec{x} = 0\vec{x} \implies \mathbf{A}\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = -2y \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix}$$

where y is a free variable.

Any vector that lies in $\text{span}\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 0$.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 2x \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$$

where x is a free variable.

Any vector that lies in $\text{span}\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ is an eigenvector corresponding to eigenvalue $\lambda = 5$.

(d) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda\vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

Solution:

Recall the definition of a matrix subspace from Note 8. A subspace $\mathbb{U}$ consists of a subset of the vector space $\mathbb{V}$ if it contains the zero vector, is closed under scalar multiplication, and is closed under vector addition.

i. Zero vector: The zero vector is contained in this set since $\mathbf{A}\vec{0} = \vec{0} = \lambda\vec{0}$.

ii. Scalar multiplication: Let $\vec{v}_1$ be a member of the set. Let $\vec{u} = \alpha\vec{v}_1$. Note that $\vec{u} \in \mathbb{R}^n$, thus a possible value of $\vec{x}$. Now, $\mathbf{A}\vec{u} = \mathbf{A}\alpha\vec{v}_1 = \alpha\mathbf{A}\vec{v}_1 = \alpha\lambda\vec{v}_1 = \lambda\vec{u}$. Hence, $\vec{u}$ is a member of the set as well and the set is closed under scalar multiplication.

iii. Vector addition: Let $\vec{v}_1$ and $\vec{v}_2$ be members of the set. Observe below that the set is closed under vector addition as well.

$$\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2 = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$$

Note that $\vec{v}_1 + \vec{v}_2$ is also a vector in $\mathbb{R}^n$, which corresponds to how $\vec{x}$ is defined in this setup.

Hence, the set defined in the question satisfies the properties of a subspace and is consequently a subspace of $\mathbb{R}^n$.

7. Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.

Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.

Solution:

Recall that we find the eigenvalues of a matrix $\mathbf{A}$ by setting the determinant of $\mathbf{A} - \lambda\mathbf{I}$ to 0.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\left((\mathbf{A} - \lambda\mathbf{I})^T\right) = \det\left(\mathbf{A}^T - \lambda\mathbf{I}\right) = 0$$

Since the two determinants are equal, the characteristic polynomials of the two matrices must also be equal. Therefore, they must have the same eigenvalues.

(b) Let a square matrix $\mathbf{A}$ have, for each row, entries that sum to one. Show that $\vec{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

Solution:

If the rows of $\mathbf{A}$ sum to one, then $\mathbf{A}\vec{1} = \vec{1}$. Therefore, the corresponding eigenvalue is $\lambda = 1$.

We can see this by simple matrix multiplication. For instance, if we look at a 3x3 matrix $\mathbf{A}$ where the first row has elements a_1, a_2, and a_3, such that $a_1 + a_2 + a_3 = 1$. We can perform the matrix multiplication, $\mathbf{A} \cdot \vec{1}$ and analyze the first element in the vector product, which is a product of the first row of $\mathbf{A}$ and $\vec{1}$.

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (1)a_1 + (1)a_2 + (1)a_3$$

$$= a_1 + a_2 + a_3 = 1$$

We see that since the first row of $\mathbf{A}$ will sum to 1, the first element in the matrix multiplication will be 1. Since each row sums to 1, every element in the matrix multiplication will be 1. Thus, the matrix multiplication $\mathbf{A}\vec{1} = \vec{1}$. By inspection, we know that the eigenvalue must be 1.

(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices have, for each column, entries that sum to 1.

Solution:

If we transpose a conservative state transition matrix $\mathbf{A}$, then the rows of $\mathbf{A}^T$ (or the columns of $\mathbf{A}$) sum to one by definition of a conservative system. Then, from part (b), we know that $\mathbf{A}^T$ has the eigenvalue $\lambda = 1$. Furthermore, from part (a), we know that the $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues, so $\mathbf{A}$ also has the eigenvalue $\lambda = 1$.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.