EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 4

This homework is due February 18, 2021, at 23:59.
Self-grades are due February 21, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.
`hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).
Submit each file to its respective assignment on Gradescope.

1. Reading Assignment

For this homework, please review Note 4, and read Notes 5, and 6. Note 4 goes into greater depth on proofs, while notes 5 and 6 provide an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps, and matrix inversion. You are always welcome and encouraged to read beyond this as well.

You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?

Solution: Each column in the state transition matrix must sum to one.

2. Feedback on your study groups

Please help us understand how your study groups are going! Fill out the following survey to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

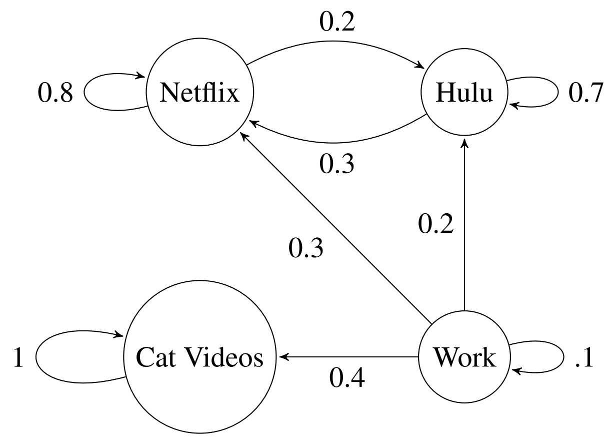
`https://forms.gle/T8y4KEMPQgAdc9Jk8`

To get full credit for this question you must (1) fill out the survey (it will record your email) and (2) indicate in your homework submission that you filled out the survey.

3. Social Media

Learning Objective: *Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.*

As a tech-savvy Berkeley student, the distractions of streaming services are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Netflix or Hulu? How do other students manage to get stuff done and balance staying up to date with the Bachelor? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if $x = 100$ students are on Netflix, in the next timestep, $0.2x$ (20) of them will click on a link and move to Hulu, and $0.8x$ (80) will remain on Netflix.



(a) Let us define $x_N[n]$ as the number of students on Netflix at timestep n, $x_H[n]$ as the number of students on Hulu at timestep n, $x_C[n]$ as the number of students watching any kind of cat vido at timestep n, and $x_W[n]$ as the number of students working at timestep n. Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix.

Hint: A transition matrix, A is the matrix that connects $\vec{x}[n]$, the vector at timestep n to the vector at timestep $n+1$, $\vec{x}[n+1]$: $\vec{x}[n+1] = A\vec{x}[n]$

Solution: Let us explicitly write the equations that we can then use to determine the state transition matrix.

$x_N[n+1] = 0.3x_H[n] + 0.3x_W[n] + 0.8x_N[n]$
$x_H[n+1] = 0.7x_H[n] + 0.2x_N[n] + 0.2x_W[n]$
$x_C[n+1] = 1x_C[n] + 0.4x_W[n]$
$x_W[n+1] = 0.1x_W[n]$

Let $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$.

We can now solve for the state transition matrix A such that:

$\vec{x}[n+1] = A\vec{x}[n]$.

A is therefore equal to:

$$\begin{bmatrix} 0.8 & 0.3 & 0 & 0.3 \\ 0.2 & 0.7 & 0 & 0.2 \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & .1 \end{bmatrix}$$

(b) There are 1500 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 700 EECS16A students on Netflix, 450 on Hulu, 200 watching Cat Videos, and 150 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

Solution:

In order to calculate the state vector at the next timestep, we can use the equation $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$. Substituting the values for A and $\vec{x}[n]$, we get the following:

$$\begin{bmatrix} 0.8 & 0.3 & 0 & 0.3 \\ 0.2 & 0.7 & 0 & 0.2 \\ 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & .1 \end{bmatrix} \begin{bmatrix} 700 \\ 450 \\ 200 \\ 150 \end{bmatrix} = \begin{bmatrix} 740 \\ 485 \\ 260 \\ 15 \end{bmatrix}$$

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

Solution:

Since each column's sum is equal to 1, the system is conservative. This means that we aren't losing students after each time step.

4. Image Stitching

Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

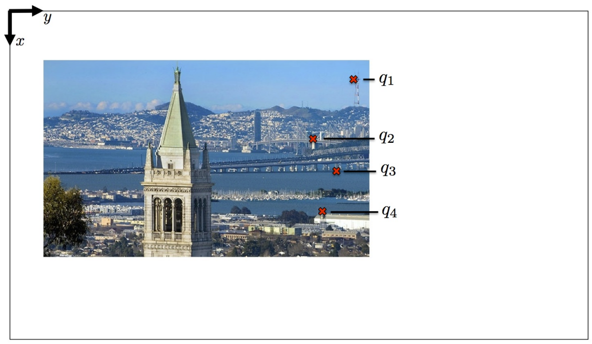
We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**

We will use vectors to represent the common points which are related by a linear transformation. Your idea is to find this linear transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image , which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 1 shows how the points $\vec{p}_1$, $\vec{p}_2$ ... in the right image are transformed to points $\vec{q}_1$, $\vec{q}_2$ ... on the left image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \tag{1}$$



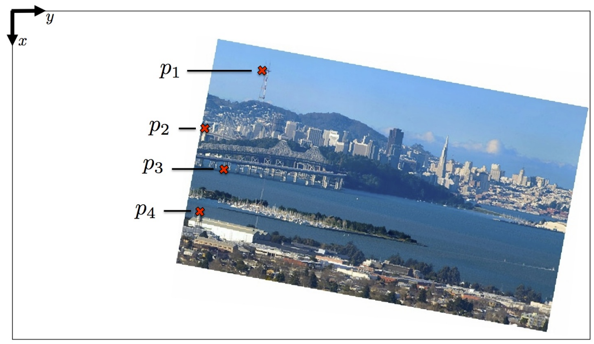


Figure 1: Two images to be stitched together with pairs of matching points labeled.

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{t}$), so that you will be able to stitch the image together.

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{t}$ transforms any vector representing a point on a image, Consider this equation similar to Equation (1),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \tag{2}$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.
We want to find out what geometric transformation(s) can be applied on $\vec{u}$ to give $\vec{v}$.

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming $\vec{u}$. Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.
What **geometric transformation(s)** might be applied to $\vec{u}$ to get $\vec{v}_1$? Choose the options that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{u}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook `prob4.ipynb`.
Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of $\vec{w}$ is geometrically transforming** $\vec{v}_1$. Choose the option that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{v}$, $\vec{w}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook `prob4.ipynb`.

Solution: Plugging in the given vectors and performing the matrix vector multiplication,

$$\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}. \tag{3}$$

It is observable that $\vec{v}_1$ is a scaled, rotated version of $\vec{u}$.
We get $\vec{v} = \vec{v}_1 + \vec{w} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. We can also see that $\vec{v}$ is a shifted version of $\vec{v}_1$.
Hence $\vec{u}$ is scaled, rotated and shifted to get $\vec{v}$.
Note: We can only use matrix transformations to scale and/rotate a vector. We cannot translate a vector through matrix transformations; instead we must use vector addition for this.

(b) Multiply Equation (1) out into **two linear equations**.
  (i) What are the known values and what are the unknown values in each equation?
  (ii) How many unknown values are there?
  (iii) How many independent equations do you need to solve for all the unknowns?
  (iv) How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns? ***Hint**: Remember that each pair of $\vec{p}$ and $\vec{q}$ will give you **two** different linear equations.*

Solution:
We can rewrite the above matrix equation as the following two scalar linear equations:

$$q_x = p_x r_{xx} + p_y r_{xy} + t_x$$
$$q_y = p_x r_{yx} + p_y r_{yy} + t_y$$

Here, the known values are each pair of points' elements: $q_x,\ q_y,\ p_x,\ p_y$, and the scaling factor of the $\vec{t}$ vector (1). The unknowns are elements of $\mathbf{R}$ and $\vec{t}$: $r_{xx},\ r_{xy},\ r_{yx},\ r_{yy},\ t_x$, and t_y. There are 6 unknowns, so we need a total of 6 equations to solve for them. For every pair of points we add, we get two more equations. Thus, we need 3 pairs of common points to get 6 equations.

(c) Use what you learned in the above two subparts to explicitly write out **just enough** linear equations of these transformations as you need to solve the system. Assume that the four pairs of points from Fig. 1 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \qquad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

Solution: From the previous part, we know that we will need six equations, as we have six unknowns. Recalling that each point provides us with two equations, we arbitrarily select the first three points:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \quad \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}$$

which yield the following system:

$$r_{xx} p_{1x} + r_{xy} p_{1y} + t_x = q_{1x} \tag{4}$$
$$r_{yx} p_{1x} + r_{yy} p_{1y} + t_y = q_{1y} \tag{5}$$
$$r_{xx} p_{2x} + r_{xy} p_{2y} + t_x = q_{2x} \tag{6}$$
$$r_{yx} p_{2x} + r_{yy} p_{2y} + t_y = q_{2y} \tag{7}$$
$$r_{xx} p_{3x} + r_{xy} p_{3y} + t_x = q_{3x} \tag{8}$$
$$r_{yx} p_{3x} + r_{yy} p_{3y} + t_y = q_{3y} \tag{9}$$

(d) Remember that we ultimately want to solve for the components of the **R** matrix and the vector $\vec{t}$; so we should try to isolate these. We can represent these unknowns in terms of a vector, $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$.

Translate your equations from the previous part into a matrix-vector formulation that will allow you to solve for $\vec{\alpha}$.

Solution: We write the system of linear equations from the previous part in matrix form.

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} & p_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{2x} & p_{2y} & 0 & 1 \\ p_{3x} & p_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{3x} & p_{3y} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

(e) In the IPython notebook `prob4.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, **R**, and vector, $\vec{t}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela's algorithm.**

Solution:

The parameters for the transformation from the coordinates of the first image to those of the second image are $\mathbf{R} = \begin{bmatrix} 1.1954 & .1046 \\ -.1046 & 1.1954 \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} -150 \\ -250 \end{bmatrix}$.

5. Mechanical Inverses

***Learning Objectives**: Matrices represent linear transformations, and their inverses represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

For each of the following values of matrix A:

i Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii **For parts (a)-(d)**, in addition to finding the inverse (if it exists), describe how the matrix **A** transforms an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

For example, if $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, then **A** could scale $\begin{bmatrix} x \\ y \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. If $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, then **A** could reflect $\begin{bmatrix} x \\ y \end{bmatrix}$ across the x axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii **For parts (a)-(d)**, if we use **A** to geometrically transform $\begin{bmatrix} x \\ y \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x \\ y \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

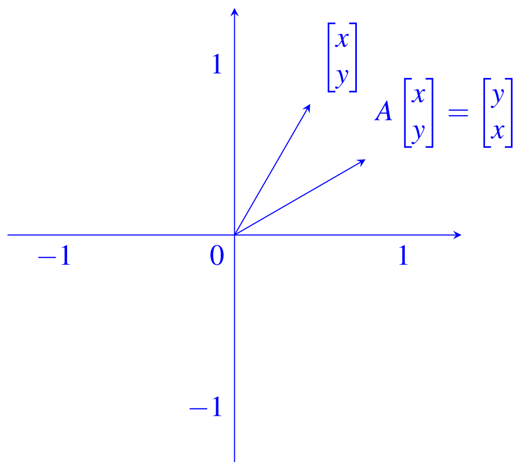
(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \text{swap } \mathrm{R}_1, \mathrm{R}_2$$

The inverse does exist.

$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



The original matrix $\mathbf{A}$ flips the x and y components of the vector. Any correct equivalent sequence of operations (such as reflecting the vector across the $x = y$ line) warrants full credit. Notice how the inverse does the exact same thing–that is, it switches the x and y components of the vector it's applied to. This makes sense–switching x and y twice on a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ gives us the same vector $\begin{bmatrix} x \\ y \end{bmatrix}$. So the transformation done by $\mathbf{A}$ is reversible.

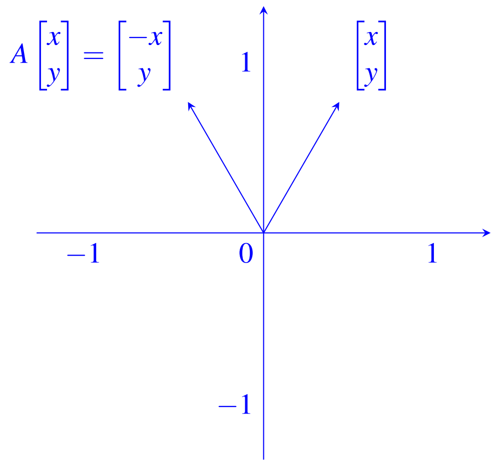
(b) $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] R_1 \leftarrow -R_1$$

The inverse does exist.

$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



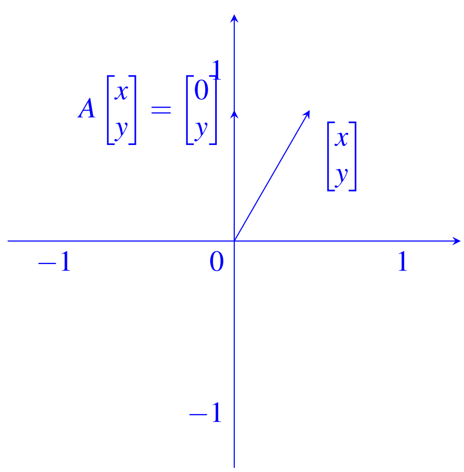
The original matrix $\mathbf{A}$ reflects the vector across the y-axis, i.e. it multiplies the vector's x-component by a factor of -1. Reflecting the vector across the y-axis again with $\mathbf{A}^{-1} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ will give you the original vector, i.e. the transformation done by $\mathbf{A}$ is reversible.

(c) $\mathbf{A} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc}0 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|cc}0 & 1 & 0 & 1\\0 & 0 & 1 & 0\end{array}\right] \text{swap } \mathrm{R}_1, \mathrm{R}_2$$

We see here that the inverse does not exist because the second row represents an inconsistent equation. Another way to see that the inverse does not exist is by realizing that the first column (and first row) of the original matrix are the zero vector, so the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.



The original matrix $\mathbf{A}$ removes the x-component of the vector it's applied to and keeps the same y-component. Graphically speaking, this matrix can be thought of as taking the "shadow" of the vector on the y-axis if you were to shine a light perpendicular to the y-axis.

Since the x-component of the vector is completely lost after the transformation, the process is not reversible.

(d) $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Assume $\cos\theta \neq 0$. *Hint:* $\cos^2\theta + \sin^2\theta = 1$.

Solution:

The inverse does exist.

$$\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1/\cos\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \cos\theta + \frac{\sin^2\theta}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 - R_1 \times \sin\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \frac{1}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \qquad \cos^2\theta + \sin^2\theta = 1$$

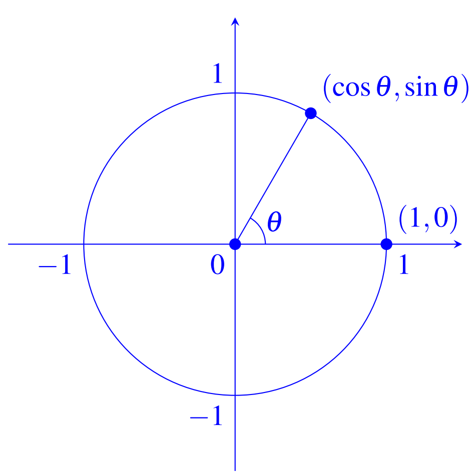
$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \qquad R_2 \leftarrow R_2 \times \cos\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \qquad R_1 \leftarrow R_1 + R_2 \times \sin\theta/\cos\theta$$

$\mathbf{A}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

The original matrix $\mathbf{A}$ is the two-dimensional rotation matrix as seen in discussion 3B. The rotation matrix rotates a vector in the counter-clockwise direction, and its inverse rotates a vector in the clockwise direction. Take the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for example:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



The inverse matrix can also be found from the rotation matrix that rotates a vector by an angle $-\theta$. The inverse matrix can also be found as follows:

$$\mathbf{A}^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

So the transformation done by $\mathbf{A}$ is a reversible process.

(e) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 - 2R_1$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \qquad R_2 \leftarrow -R_2/2$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \qquad R_1 \leftarrow R_1 - R_2$$

Inverse exists: $\mathbf{A}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$.

(f) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 4 & 4 & -1 & 0 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - 2R_2$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the second and third columns are equal, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(g) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1 \times -1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_2/2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \qquad R_3 \leftarrow 2R_3/3; R_2 \rightarrow R_2/2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \qquad R_2 \leftarrow R_2 + R_3/2; R_1 \leftarrow R_1 - R_3/2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \qquad R_1 \leftarrow R_1 + R_2$$

Inverse exists: $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(h) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] R_3 \leftarrow R_3 + R_2$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the third column is the negative of the second column, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(i) **(OPTIONAL)**

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

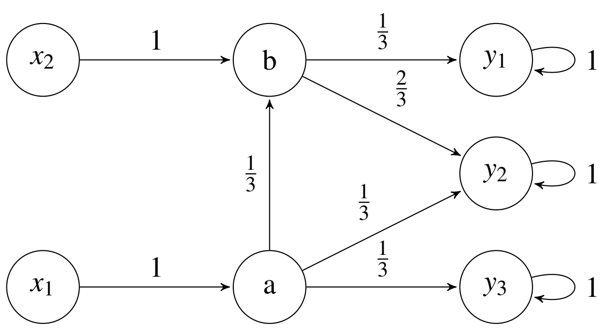
Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix? Hint 2: We're reasonable people!

Solution:

Inverse does not exist because $\text{column}_1 + \text{column}_2 + \text{column}_3 = \text{column}_4$, which means that the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

6. Functional Pumps

Let us model a linear function $f(x_1, x_2)$ with outputs y_1, y_2, y_3 as a pump given by the diagram below.



(a) Write y_1, y_2, y_3 in terms of x_1, x_2.

Solution:

$$y_1 = \frac{1}{9}x_1 + \frac{1}{3}x_2$$

$$y_2 = \frac{5}{9}x_1 + \frac{2}{3}x_2$$

$$y_3 = \frac{1}{3}x_1$$

(b) Using the relations you created above, we can write

$$f(x_1, x_2) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the matrix **A**

Solution:

$$\mathbf{A} = \begin{bmatrix} \frac{1}{9} & \frac{1}{3} \\ \frac{5}{9} & \frac{2}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$

(c) Write the state transition matrix for the above state transistion diagram. Use the state vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ a \\ b \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Think about what steps need to be taken to reduce it to the form in part b.

Solution: We should first begin by identifying the unknown values. In this case, it is the value at each node–since we have 7 such nodes, we can write seven equations that represent the change in the state vector from timestep n to $n+1$, as follows:

$$0x_1[n] + 0x_2[n] + 0a[n] + 0b[n] + 0y_1[n] + 0y_2[n] + 0y_3[n] = x_1[n+1] \quad (10)$$
$$0x_1[n] + 0x_2[n] + 0a[n] + 0b[n] + 0y_1[n] + 0y_2[n] + 0y_3[n] = x_2[n+1] \quad (11)$$
$$1x_1[n] + 0x_2[n] + 0a[n] + 0b[n] + 0y_1[n] + 0y_2[n] + 0y_3[n] = a[n+1] \quad (12)$$
$$0x_1[n] + 1x_2[n] + \frac{1}{3}a[n] + 0b[n] + 0y_1[n] + 0y_2[n] + 0y_3[n] = b[n+1] \quad (13)$$
$$0x_1[n] + 0x_2[n] + 0a[n] + \frac{1}{3}b[n] + 1y_1[n] + 0y_2[n] + 0y_3[n] = y_1[n+1] \quad (14)$$
$$0x_1[n] + 0x_2[n] + \frac{1}{3}a[n] + \frac{2}{3}b[n] + 0y_1[n] + 1y_2[n] + 0y_3[n] = y_2[[n+1] \quad (15)$$
$$0x_1[n] + 0x_2[n] + \frac{1}{3}a[n] + 0b[n] + 0y_1[n] + 0y_2[n] + 1y_3[n] = y_3[[n+1] \quad (16)$$
$$(17)$$

we can assemble our unknowns into a vector, $\vec{\alpha} = \begin{bmatrix} x_1 \\ x_2 \\ a \\ b \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Recall that a transition matrix, T is defined by the equation: $s[n+1] = Ts[n]$, with $s[i]$ representing the state at time i. Reformating the system above into matrix vector form yields:

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 \end{bmatrix}$$

7. Matrix Operations Practice

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 6 \\ 1 & 2 \end{bmatrix} B = \begin{bmatrix} 8 & 6 & 4 \\ 1 & 3 & 5 \end{bmatrix} C = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} D = \begin{bmatrix} 3 & 10 \\ 7 & 2 \end{bmatrix}$$

(a) Compute the following matrix multiplications.

i AB

ii BA

iii AD

Solution:

i $AB = \begin{bmatrix} 26 & 24 & 22 \\ 46 & 48 & 50 \\ 10 & 12 & 14 \end{bmatrix}$

ii $BA = \begin{bmatrix} 58 & 60 \\ 23 & 30 \end{bmatrix}$

iii $AD = \begin{bmatrix} 23 & 34 \\ 57 & 62 \\ 17 & 14 \end{bmatrix}$

(b) Compute the following transposes.

i A^T

ii B^T

iii C^T

iv D^T

Solution:

i $A^T = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 6 & 2 \end{bmatrix}$

ii $B^T = \begin{bmatrix} 8 & 1 \\ 6 & 3 \\ 4 & 5 \end{bmatrix}$

iii $C^T = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

iv $D^T = \begin{bmatrix} 3 & 7 \\ 10 & 2 \end{bmatrix}$

(c) Show that for general matrices $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{p \times q}$

$$(AB)C = A(BC)$$

.

Solution: Let general entries of the matrices A, B, C be A_{ij}, B_{jk}, and C_{kl} respectively. Also note that:

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}$$

$$(BC)_{jl} = \sum_{k=1}^{p} B_{jk} C_{kl}$$

$$\begin{aligned}((AB)C)_{il} &= \sum_{k=1}^{p} (AB)_{ik} C_{kl} \\ &= \sum_{k=1}^{p} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{kl} \\ &= \sum_{k=1}^{p} \sum_{j=1}^{n} (A_{ij} B_{jk}) C_{kl}\end{aligned}$$

Notice that A_{ij}, B_{jk}, C_{kl} are all scalars, and *scalar* multiplication is commutative, and if there is a constant in a summation (i.e. something which is not dependent on the indices in the summation), it can be pulled out of the sum.

$$\begin{aligned}&= \sum_{j=1}^{n} \sum_{k=1}^{p} A_{ij} (B_{jk} C_{kl}) \\ &= \sum_{j=1}^{n} A_{ij} \left(\sum_{k=1}^{p} B_{jk} C_{kl} \right)\end{aligned}$$

The above step was possible because neither i nor j appear in the second sum, which is over k.

$$\begin{aligned}&= \sum_{j=1}^{n} A_{ij} (BC)_{jl} \\ &= (A(BC))_{il}\end{aligned}$$

These two simplifying steps are done because the product fits the form of matrix multiplication. Feel free to try it out on two simple vectors and you will see that this summation succinctly captures matrix/vector multiplication.

(d) Show that for general matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$:

$$(AB)^T = B^T A^T$$

Hint: For two vectors, $\vec{a}$ and $\vec{b}$, $\vec{a}^T \vec{b} = \vec{b}^T \vec{a}$

Solution:

Method 1

Note $(AB)_{ji}$ indicates the entry of the matrix on the j-th row and i-th column.
With $\vec{a}_j^T$ representing the j-th row of matrix A and $\vec{b}_i$ representing the i-th column of B:

$$\begin{aligned}((AB)^T)_{ij} &= (AB)_{ji} \\ &= \vec{a}_j^T \vec{b}_i \\ &= \vec{b}_i^T \vec{a}_j \\ &= (B^T A^T)_{ij}\end{aligned}$$

$$\therefore (AB)^T = B^T A^T$$

Method 2

Let $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ and $\{\vec{b}_1, \vec{b}_2, ..., \vec{b}_p\}$ represent the columns of A and B respectively.
With this take note of a substitution that will be helpful later - that is for all i:

$$\begin{aligned}
(A\vec{b}_i)^T &= (\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n\end{bmatrix}\vec{b}_i)^T \\
&= (b_{i1}\vec{a}_1 + b_{i2}\vec{a}_2 + \dots + b_{in}\vec{a}_n)^T \\
&= b_{i1}\vec{a}_1^T + b_{i2}\vec{a}_2^T + \dots + b_{in}\vec{a}_n^T \\
&= \begin{bmatrix}b_{i1} & b_{i2} & \dots & b_{in}\end{bmatrix}\begin{bmatrix}\vec{a}_1^T \\ \vec{a}_2^T \\ \dots \\ \vec{a}_n^T\end{bmatrix} \\
&= \vec{b}_i^T A^T
\end{aligned}$$

$$\begin{aligned}
(AB)^T &= (A\begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_p\end{bmatrix})^T \\
&= \begin{bmatrix}A\vec{b}_1 & A\vec{b}_2 & \dots & A\vec{b}_p\end{bmatrix}^T \\
&= \begin{bmatrix}(A\vec{b}_1)^T \\ (A\vec{b}_2)^T \\ \dots \\ (A\vec{b}_p)^T\end{bmatrix} \\
&= \begin{bmatrix}\vec{b}_1^T A^T \\ \vec{b}_2^T A^T \\ \dots \\ \vec{b}_p^T A^T\end{bmatrix} \\
&= B^T A^T
\end{aligned}$$

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

