EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 3

This homework is due February 11th, 2022, at 23:59.
Self-grades are due February 18th, 2022, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Reading Assignment** For this homework, please read Note 3 and 11A. Note 3 overview of span and an introduction to thinking about and writing proofs. Note 11A will give you an introduction to circuits.

   Please answer the following questions:

   (a) What does span of a set of vectors mean?

   (b) How can you check if a particular vector is in the span of a set of vectors?

   (c) Given that $\vec{b} \in \text{span}\{\ \vec{a}_1, \vec{a}_2, \vec{a}_3\ \}$ and $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are column vectors of $\mathbf{A}$, which *one* of the following statements does not make sense:
   i. $\vec{b}$ is in the span of matrix $\mathbf{A}$
   ii. $\vec{b}$ is in the range of $\mathbf{A}$
   iii. $\vec{b}$ is in the column space of $\mathbf{A}$

   Solution:

   (a) See Note 3.3 for definition of span of a set of vectors.

   (b) We can check that a particular vector is in the span of a set of vectors if we can write that particular vector as a linear combination of the set of vectors.

   (c) There is no such thing as a span of a matrix, only a span of a set of vectors (or column vectors within the matrix). Statement i is wrong.

2. **Vectors in the Span**

   Learning Goal: *Practice determining whether a vector is in the span of a set of vectors.*

   Determine whether a vector $\vec{v}$ is in the span of the given set of vectors. If it is in the span of given set, write $\vec{v}$ as a linear combination of given set of vectors (you will need to find the scalar coefficients in the linear combination).

   (a) $\vec{v} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$ and $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

   Solution: We see that $\vec{v}$ is a scaled multiple of one of the vectors. $\vec{v} = 2\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ thus $\vec{v}$ is in the span of the set of vectors.

(b) $\vec{v} = \begin{bmatrix}-1\\0\\-1\\0\\1\end{bmatrix}$ and $\left\{\begin{bmatrix}-1\\1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\-2\\-1\end{bmatrix}\right\}$

Solution:

We want to write the following:

$$\vec{v} = a\begin{bmatrix}-1\\1\\0\\-1\\1\end{bmatrix} + b\begin{bmatrix}1\\2\\3\\-2\\-1\end{bmatrix}$$

We can use Gaussian Elimination to find a and b:

$$\left[\begin{array}{cc|c}-1 & 1 & -1\\1 & 2 & 0\\0 & 3 & -1\\-1 & -2 & 0\\1 & -1 & 1\end{array}\right] \xrightarrow{R_1 \leftarrow -R_1} \left[\begin{array}{cc|c}1 & -1 & 1\\1 & 2 & 0\\0 & 3 & -1\\-1 & -2 & 0\\1 & -1 & 1\end{array}\right] \xrightarrow[R_4 \leftarrow R_4+R_1]{R_2 \leftarrow R_2-R_1} \left[\begin{array}{cc|c}1 & -1 & 1\\0 & 3 & -1\\0 & 3 & -1\\0 & -3 & 1\\1 & -1 & 1\end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2/3]{R_5 \leftarrow R_5-R_1} \left[\begin{array}{cc|c}1 & -1 & 1\\0 & 1 & -\frac{1}{3}\\0 & 3 & -1\\0 & -3 & 1\\0 & 0 & 0\end{array}\right] \xrightarrow[R_4 \leftarrow R_4+3R_2]{R_3 \leftarrow R_3-3R_2} \left[\begin{array}{cc|c}1 & -1 & 1\\0 & 1 & -\frac{1}{3}\\0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{array}\right]$$

This gives us a = $\frac{2}{3}$ and b = -$\frac{1}{3}$ and we can write: $\vec{v} = \frac{2}{3}\begin{bmatrix}-1\\1\\0\\-1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\2\\3\\-2\\-1\end{bmatrix}$.

(c) $\vec{v} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$ and $\left\{\begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\4\\-1\end{bmatrix}\right\}$

Solution:

Using the same method as part (b):

$$\left[\begin{array}{ccc|c}2 & 0 & 2 & 0\\2 & 1 & 4 & -1\\0 & 1 & -1 & 1\end{array}\right] \xrightarrow{R_1 \leftarrow R_1/2} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\2 & 1 & 4 & -1\\0 & 1 & -1 & 1\end{array}\right] \xrightarrow{R_2 \leftarrow R_2-2R_1} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 2 & -1\\0 & 1 & -1 & 1\end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3-R_2} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 2 & -1\\0 & 0 & -3 & 2\end{array}\right] \xrightarrow{R_3 \leftarrow R_3/-3} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 2 & -1\\0 & 0 & 1 & -\frac{2}{3}\end{array}\right] \xrightarrow[R_1 \leftarrow R_1-R_3]{R_2 \leftarrow R_2-2R_3} \left[\begin{array}{ccc|c}1 & 0 & 0 & \frac{2}{3}\\0 & 1 & 0 & \frac{1}{3}\\0 & 0 & 1 & -\frac{2}{3}\end{array}\right]$$

We can write $\vec{v} = \frac{2}{3}\begin{bmatrix}2\\2\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}2\\4\\-1\end{bmatrix}$

(d) $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \right\}$

Solution: Note that $\vec{v}$ is $\vec{0}$, the zero vector since all its values are zeros. We can write a trivial linear combination using zeros as scalars:

$$0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

There will always exist a zero vector that exists in the span of a set of vectors.

3. Span Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills. Refer to problem 3 in Discussion 2B for a similar proof with span.*

(a) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

Solution:
Suppose $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars a_i:

$$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n$$

We can change the scalar values to adjust for the combined vectors. Thus, we have shown that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\vec{r} = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are the same.

(b) Consider the span of the set $(\vec{v}_1, ..., \vec{v}_n, \vec{u})$. Suppose $\vec{u}$ is in the span of $\{\vec{v}_1, ..., \vec{v}_n\}$. Then, show that any vector $\vec{r}$ in $span\{\vec{v}_1, ..., \vec{v}_n, \vec{u}\}$ is in $span\{\vec{v}_1, ..., \vec{v}_n\}$.

Solution: From the first sentence of the question, by definition of span, we know that any vector $\vec{r}$ in $span\{\vec{v}_1, ..., \vec{v}_n, \vec{u}\}$ can be written $\vec{r} = k\vec{u} + a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$. Using the summation symbol, we can also write $\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$.
From the second sentence of the question, since $\vec{u}$ is in the span of $\{\vec{v}_1, ..., \vec{v}_n\}$, we can write $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + ...b_n\vec{v}_n$ or $\sum_{i=1}^{n} b_i\vec{v}_i$. Now that we have an expression for $\vec{u}$, let's substitute it into the previous expression.

$$\vec{r} = k\vec{u} + \sum_{i=1}^{n} a_i\vec{v}_i$$

$$\vec{r} = k(\sum_{i=1}^{n} b_i\vec{v}_i) + \sum_{i=1}^{n} a_i\vec{v}_i$$

Finally, gathering up coefficients, we get:

$$\vec{r} = \sum_{i=1}^{n} (k * b_i + a_i)\vec{v}_i,$$

so this arbitrary vector $\vec{r}$ is also in $span\{\vec{v}_1, ..., \vec{v}_n)\}$.
Intuitively, $\vec{u}$ is redundant, so we can safely remove it without reducing our span.

4. Basic Circuit Components

Learning Objectives: *Review basics of cucuit components and current-voltage relationships*

(a) Fill in the units for the following quanities:

Quantity	**Symbol**	**Units**
Voltage	V	
Current	I	
Resistance	R	

Solution:

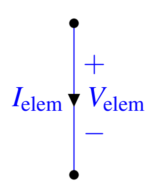
Quantity	**Symbol**	**Units**
Voltage	V	Volts (V)
Current	I	Amperes (A)
Resistance	R	Ohms (Ω)

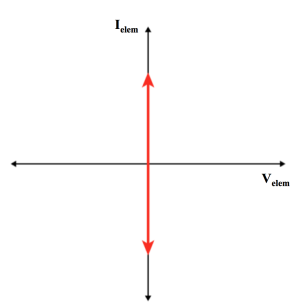
(b) What is the voltage across a **short circuit (wire)**? What is the current going through it? Draw the symbol and sketch the IV relationship.

Solution: A wire is an ideal connection with zero voltage across it. The current through the wire is determined by the rest of the circuit.

IV Relationship

Symbol



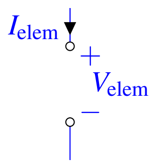


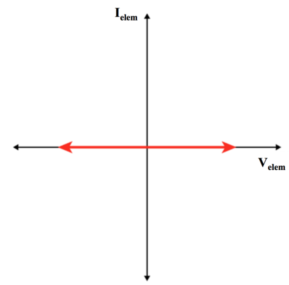
(c) What is the voltage across an **open circuit**? What is the current going through it? Draw the symbol and sketch the IV relationship.

Solution: There is no current going through an open circuit. The voltage potential across an open circuit is determined by the rest of the circuit.

IV Relationship

Symbol



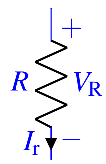


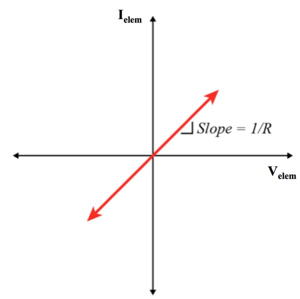
(d) What is the relationship between voltage and current for a **resistor**? Draw the symbol and IV relationship.

Solution: The relationship is described by Ohm's Law: $V_R = I_R R$

IV Relationship

Symbol

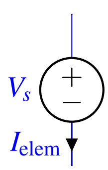


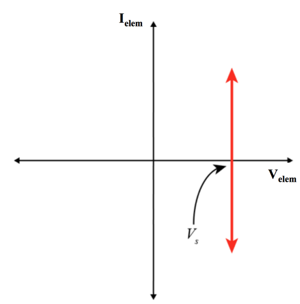


(e) What is the voltage across a **voltage source** V_s? What is the current going through it? Draw the symbol and sketch the IV relationship. **Solution:** The voltage across the voltage source is always equal to the source value, V_s. The current through a voltage source is determined by the rest of the circuit.

IV Relationship

Symbol

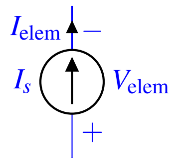




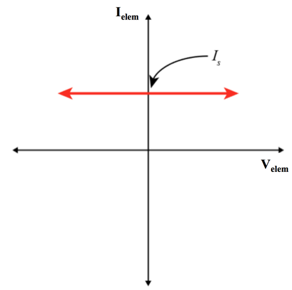
(f) What is the voltage across a **current source** I_s? What is the current going through it? Draw the symbol and sketch the IV relationship.

Solution: The voltage across a current source is determined by the rest of the circuit. The current through a current source is always equal to the source value I_s.

Symbol



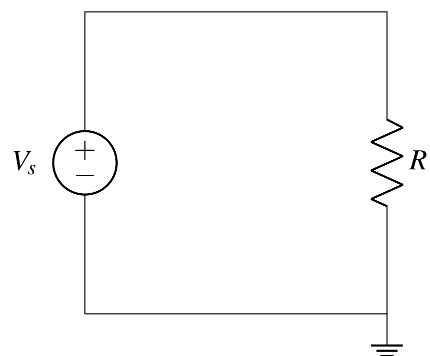
IV Relationship



5. Ohm's Law

Learning Objectives: *Practice implementing Ohm's Law in basic circuits.*

(a) Take the circuit diagram below.
V_s = 5 V and R = 10 Ω.

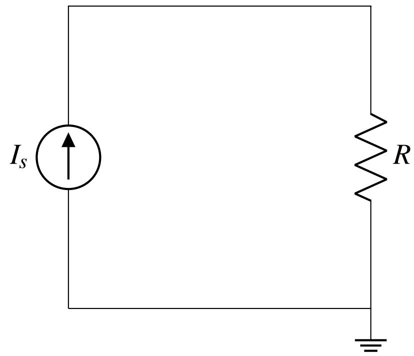


i. How many nodes are in this circuit?
ii. What is the potential at ground?
iii. What is V_R, the voltage across R?
iv. What is I_R, the current through R?

Solution:

i. There are two nodes in this circuit, one at the top and one at the bottom (ground node).
ii. The ground node is always referenced as 0 potential.
iii. $V_R = V_s = 5$ V.
iv. Using Ohm's Law, $I_R = \frac{V_R}{R} = \frac{5V}{10\Omega} = \frac{1}{2}A = 500$ mA.

(b) Now switch the voltage source with a current source.
I_s = 0.002 A = 2 mA and R = 10,000,000 Ω = 10 MΩ.

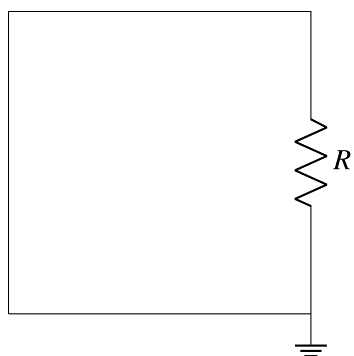


i. How many nodes are in this circuit?
ii. What is the potential at ground?
iii. What is I_R, the current through R?
iv. What is V_R, the voltage across R?

Solution:

i. There are two nodes in this circuit, one at the top and one at the bottom (ground node).
ii. The ground node is always referenced as 0 potential.
iii. $I_R = I_s = 2$ mA.
iv. Using Ohm's Law, $V_R = I_R R = (2 \text{ mA})(10 \text{ M } \Omega) = 20$ kV.

(c) Now short the circuit.



i. How many nodes are in this circuit?
ii. What is V_R, the voltage across R?
iii. What is I_R, the current through R?

Solution:

i. Since the resistor is shorted, there is only one node.
ii. There is no voltage because both sides of the resistor is connected to ground. There is no potential drop across the resistor.
iii. Since there is no potential across the resistor, there is no current flowing through the resistor.

6. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.