EECS 16A Spring 2022 — Designing Information Devices and Systems I — Homework 2

This homework is due February 4th, 2022, at 23:59.
This homework is due February 7th, 2022, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please read Notes 2A, and 2B. They will provide an overview of Gaussian elimination, vectors, and matrices. You are always welcome and encouraged to read beyond this as well.
Please answer the following questions: How can Gaussian elimination help you determine if there are no solutions to a particular system of equations? How can Gaussian elimination help you determine if a particular system has a unique solution? How about an infinite number of solutions? Does a row of zeros always mean there are infinite solutions?

Solution: This is an example solution and you should give yourself full credit for any reasonable answers.

There are three situations that can result following Gaussian elimination. We assume that the system of equations has n unknowns.

Case 1: No Solution
If the augmented matrix has any rows with all-zero variable coefficients but a nonzero result (corresponding to $0 = a$ where $a \neq 0$), then there is no solution.

Case 2: Unique Solution
If the augmented matrix has n non-zero rows (i.e. n entries in pivot position) and any rows with all-zero variable coefficients also have a zero result (corresponding to $0 = 0$), then there is a unique solution.

Case 3: Infinite Solutions
If the augmented matrix has fewer than n non-zero rows (i.e. fewer than n entries in pivot position) and any rows with all-zero variable coefficients also have a zero result (corresponding to $0 = 0$), then there are infinite solutions. A row of zeros does not always mean infinite number of solutions to the system. If there is system with N equations and K unknowns where $N > K$, then rows of zeros mean that $N - K$ equations are linear combinations of the first K equations. In this case a row of zeros does not mean an infinite number of solutions.

2. Image Masks

Learning Objective: *Learn to setup imaging problems with matrices.*

For these word problems, you only need to setup the problem with Gaussian elimination or matrix-vector notation. Of course, you may solve for practice, but no additional credit is awarded.

Solution: Full credit is awarded for setting up the augmented matrix or matrix-vector notation correctly.

After your first EECS16A lecture, you decide to try to build a single-pixel camera. You want to take a 2x2 image, i.e. 4 tiles, and based on the first lecture, you choose to take 4 measurements. Recall that each measurement is the sum of the illuminated tiles. For each measurement, you will use a different mask.

(a) Initially, you want to illuminate only one tile for each measurement. That is, you will first illuminate x_1, then you will illuminate x_2, etc. The outputs of your 4 measurements are y_1, y_2, y_3, and y_4 respectively. The 4 measurements you take are shown in Figure **??**. Explicitly setup the matrix problem for this in the $A\vec{x} = \vec{b}$ form.

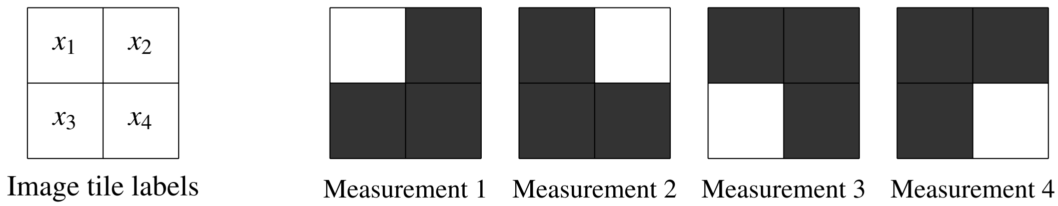


Figure 1: Four image masks.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 1 & 0 & y_3 \\ 0 & 0 & 0 & 1 & y_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

(b) While setting up your code to create the masks, you forget to turn off the illuminated tiles from the previous measurement. As a result, measurement one contains x_1, measurement two contains $x_1 + x_2$, etc. The outputs of your 4 measurements are z_1, z_2, z_3, and z_4 respectively. The 4 measurements you take are shown in Figure **??**. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.

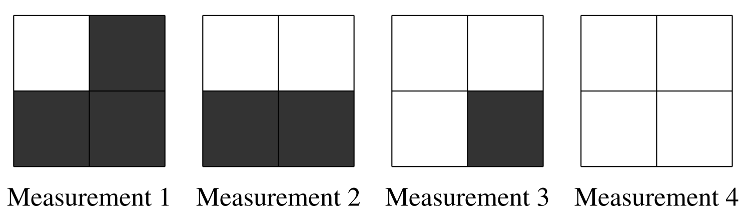


Figure 2: Four image masks.

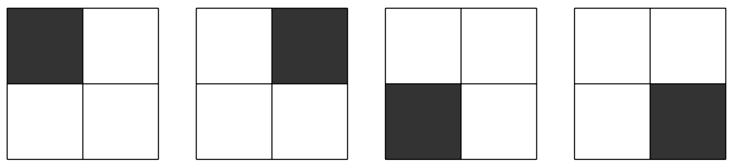
Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & z_1 \\ 1 & 1 & 0 & 0 & z_2 \\ 1 & 1 & 1 & 0 & z_3 \\ 1 & 1 & 1 & 1 & z_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

(c) Your friend is also building their own single pixel camera. However, they make a different mistake in their code and during each measurement, instead of lighting up one tile, you light up the other 3 tiles instead. That is, instead of measuring x_1, they measure $x_2 + x_3 + x_4$. The output of the 4 measurements are w_1, w_2, w_3, and w_4. The 4 measurements from their setup are shown in Figure **??**. Explicitly setup the matrix problem for this in $A\vec{x} = \vec{b}$ form.



Measurement 1 Measurement 2 Measurement 3 Measurement 4

Figure 3: Four image masks.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & w_1 \\ 1 & 0 & 1 & 1 & w_2 \\ 1 & 1 & 0 & 1 & w_3 \\ 1 & 1 & 1 & 0 & w_4 \end{array}\right]$$

The matrix-vector setup looks like this:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

3. Vector-Vector Multiplication

Learning Objective: *Practice evaluating vector-vector multiplication.*

For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

(a) i. $\vec{x}^T \cdot \vec{z}$

Solution: This is invalid. $\vec{x}^T$ is an $1 \times N$ vector meaning that it has 1 rows and N column but $\vec{z}$ is an $M \times 1$ vector meaning that is has M rows and 1 column. Since $\vec{x}^T$ does not have the same number of columns as $\vec{z}$ has rows there is no solution.

ii. $\vec{x} \cdot \vec{x}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{x}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{x}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{x}^T$.

iii. $\vec{x} \cdot \vec{y}^T$

Solution:

$$N \times N$$

$\vec{x}$ has N row and 1 columns and $\vec{y}^T$ has 1 row and N columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{y}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{y}^T$.

iv. $\vec{x} \cdot \vec{z}^T$

Solution:

$$N \times M$$

$\vec{x}$ has N row and 1 columns and $\vec{z}^T$ has 1 row and M columns. Since the number of columns of $\vec{x}$ is the same as the number of rows of $\vec{z}^T$, there is a solution. The solution would have the dimensions of the number of rows of $\vec{x}$ times the number of columns of $\vec{z}^T$.

4. Multiply the Matrices

Learning Objective: *Practice evaluating matrix-matrix multiplication.*

(a) We have two matrices $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}$ is a 3×2 matrix and $\mathbf{B}$ is a 2×4 matrix. Would the multiplication $\mathbf{AB}$ be a valid operation? If yes, what do you expect the dimensions of $\mathbf{AB}$ to be?

Solution:

This is a valid matrix-matrix multiplication as the number of columns in $\mathbf{A}$ matches the number of rows in $\mathbf{B}$. The resulting matrix will have dimensions 3×4, i.e. it will have the same number of rows as $\mathbf{A}$ and the same number of columns as $\mathbf{B}$.

(b) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

Compute $\mathbf{BA}$ too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \times 1 + 0 \times -3 & 1 \times 2 + 0 \times 0 & 1 \times -1 + 0 \times 2 & 1 \times 0 + 0 \times -1 \\ 2 \times 1 + 1 \times -3 & 2 \times 2 + 1 \times 0 & 2 \times -1 + 1 \times 2 & 2 \times 0 + 1 \times -1 \\ 0 \times 1 + 1 \times -3 & 0 \times 2 + 1 \times 0 & 0 \times -1 + 1 \times 2 & 0 \times 0 + 1 \times -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -3 & 0 & 2 & -1 \end{bmatrix}
\end{aligned}$$

BA does not exist since the number of columns in **B** is not equal to the number of rows in **A**.

(c) Compute **AB** by hand, where **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 3\times-2+21\times-1+9\times3 & 3\times4+21\times2+9\times-6 \\ -1\times-2+14\times-1+4\times3 & -1\times4+14\times2+4\times-6 \\ 7\times-2+-8\times-1+2\times3 & 7\times4+-8\times2+2\times-6 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

BA does not exist since the number of columns in **B** is not equal to the number of rows in **A**.

5. Filtering Out The Troll

Learning Goal: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

You were attending the 16A lecture the day before the first exam, and decided to record it using two directional microphones (one microphone receives sound from the x direction and the other from the y direction), because you don't trust the Zoom overlords. However, someone (we have *no* idea who) in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

The diagram shown in Figure **??** shows the locations of the speaker, the troll, and you and your two microphones (at the origin).

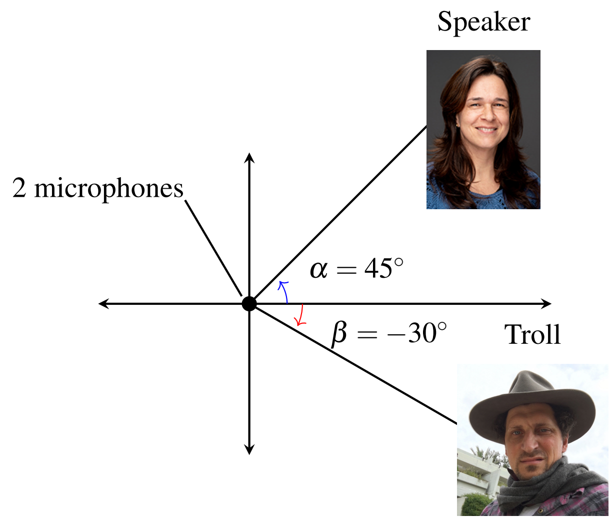


Figure 4: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the x-axis (in our case, these angles are 45°and $-30°$, labeled as α and β, respectively). The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha) \cdot \vec{s} + \cos(\beta) \cdot \vec{r} \tag{1}$$

$$\vec{m}_2 = \sin(\alpha) \cdot \vec{s} + \sin(\beta) \cdot \vec{r} \tag{2}$$

where α and β are the angles at which the professor and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the professor and the troll respectively.

(a) Plug in $\alpha = 45° = \frac{\pi}{4}$ and $\beta = -30° = -\frac{\pi}{6}$ to Equations **??** and **??** to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

Solution:

$$\begin{aligned}
\vec{m}_1 &= \cos\left(\frac{\pi}{4}\right) \cdot \vec{s} + \cos\left(-\frac{\pi}{6}\right) \cdot \vec{r} \\
&= \frac{1}{\sqrt{2}} \cdot \vec{s} + \frac{\sqrt{3}}{2} \cdot \vec{r} \\
\vec{m}_2 &= \sin\left(\frac{\pi}{4}\right) \cdot \vec{s} + \sin\left(-\frac{\pi}{6}\right) \cdot \vec{r} \\
&= \frac{1}{\sqrt{2}} \cdot \vec{s} - \frac{1}{2} \cdot \vec{r}
\end{aligned}$$

(b) Solve the system using any convenient method you prefer from the earlier part to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c \cdot \vec{m}_1 + k \cdot \vec{m}_2$ (where c and k are scalars). What are the values of c and k?

Solution: Solving the system of linear equations yields

$$\vec{s} = \frac{\sqrt{2}}{1+\sqrt{3}} \cdot \left(\vec{m}_1 + \sqrt{3}\vec{m}_2\right).$$

Therefore, the values are $c = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $k = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that $\vec{r} = \frac{2}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2)$. Substituting b back into the second equation and multiplying through by $\sqrt{2}$ gives that $\vec{s} = \sqrt{2}(\vec{m}_2 + \frac{1}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45°) = \cos(45°)$. So we know that the result of subtracting one microphone recording from the other results in only the troll's contribution. Once we have the troll contribution, we can remove it and obtain the professor's sole content.

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol2.ipynb`. The speaker (Professor Arias) is discussing Node Voltage Analysis, a circuit analysis algorithm you will learn in Module 2, and the audio is taken from a lecture in which Professor Lustig was being *particularly* disruptive.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

6. Linearity

In this question, we will explore in further detail what exactly it means for a function to be linear. For each of the following, please specify the values of a (a real number) for which the function is linear. Here, x and y are variables.

(a)

$$f(x,y) = (3-a)x + 2ay$$

Solution: Recall the definition of linearity, from Note 1:

A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is *linear* if for all real-valued $\alpha, \beta, y_1, \ldots, y_n, z_1, \ldots, z_n$, the following identity holds:

$$f(\alpha y_1 + \beta z_1, \alpha y_2 + \beta z_2, \ldots, \alpha y_n + \beta z_n) = \alpha f(y_1, \ldots, y_n) + \beta f(z_1, \ldots, z_n). \tag{3}$$

For this question, the equivalent theorem will be more useful: If $f : \mathbb{R}^n \to \mathbb{R}$ is linear, then there exist coefficients $c_1, c_2, \ldots, c_n$ (i.e., real constants, not depending on the input to the function) such that

$$f(x_1, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \quad \text{for all } x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}. \tag{4}$$

We can see that our variables are x and y (analogs of $x_1 \ldots, x_n$), and their coefficients are $(3-a)$ and $2a$, respectively. Any real value of a will make these coefficients real numbers, and the function linear.

(b)

$$f(x,y) = a^2 x + 8y$$

Solution: Following a similar line of logic as in part a, the coefficients of our variables are a^2 and 1. Any real value of a results in a real value of a^2, and hence real values for all the c_i.

(c)

$$f(x,y) = y + axy - 3x$$

Solution: Notice that this equation is the first where we have a term that multiplies two of our variables. Referencing our theorem, we see that this is not allowed–the function may only be a summation of variables multiplied by a scalar. In other words, we need to eliminate the axy term; the only way to do so is if a is zero.

(d)

$$f(x,y) = (x + ay)^2$$

Solution: If we expand this function, we would get:

$$f(x,y) = x^2 + 2axy + a^2 y^2 \tag{5}$$

Once again, we have nonlinear terms; however, the x^2 term cannot be eliminated, so there are no values of a for which this equation is linear.

7. Gaussian Elimination

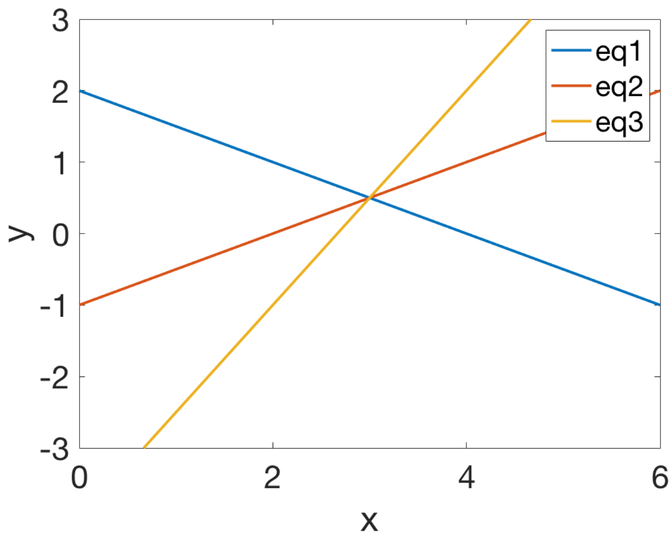
Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Plot the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \tag{1}$$
$$2x - 4y = 4 \tag{2}$$
$$3x - 2y = 8 \tag{3}$$

Solution:
The three lines intersect at the point (3,0.5).



ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

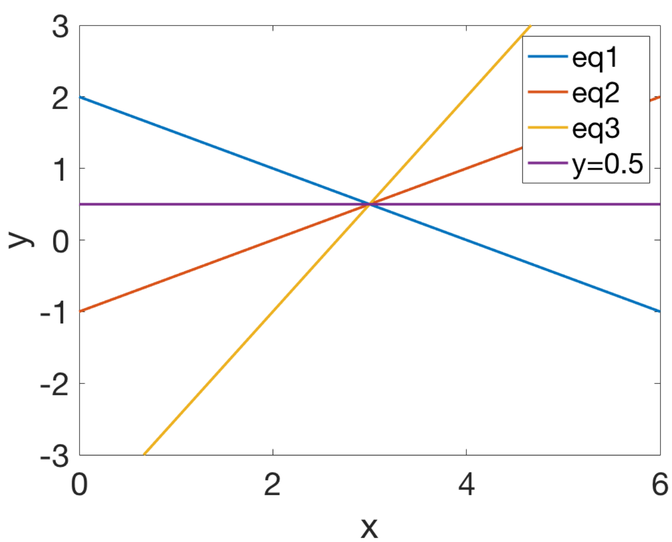
Solution: We start with the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & -4 & 4 \\ 3 & -2 & 8 \end{array}\right]$$

We then eliminate x from the second equation by subtracting $2\times$Row 1 from Row 2:

$$\text{Row 2: subtract } 2\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

So equation 2 becomes $-8y = -4$, which is equivalent to $y = 0.5$. You will notice that the line $y = 0.5$ intersects with the three lines you drew previously.



iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

Solution:

We continue from the previous part, where we had the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

and take the following steps to complete Gaussian elimination:

$$\text{Row 3: subtract } 3\times\text{Row 1} \implies \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 0 & -8 & -4 \end{array}\right]$$

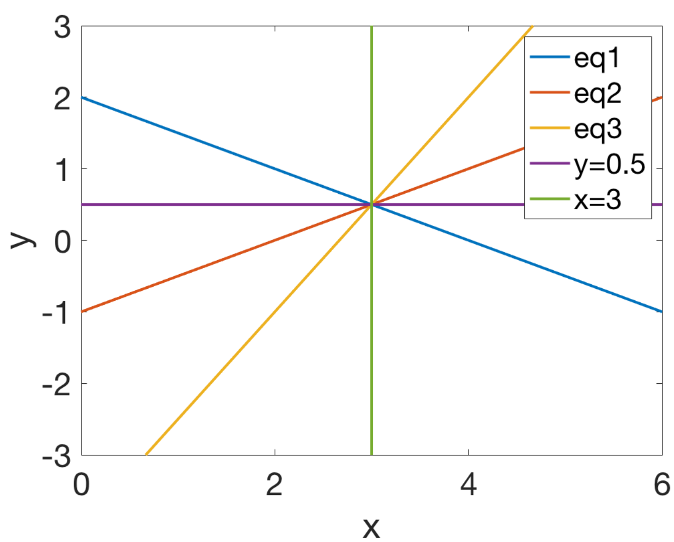
$$\text{Row 2: divide by } -8 \implies \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 3: subtract } -8\times\text{Row 2} \implies \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \implies \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, we end up with the solution $x = 3$ and $y = 0.5$.

Plotting the new equation $x = 3$ on the same graph as before, we see that all five lines intersect at the same point (3,0.5).



(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$\begin{aligned} x+2y+5z&=3\\ x+12y+6z&=1\\ 2y+z&=4\\ 3x+16y+16z&=7 \end{aligned}$$

Solution:

Writing the system in augmented matrix form we get the following:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 3\\ 1 & 12 & 6 & 1\\ 0 & 2 & 1 & 4\\ 3 & 16 & 16 & 7 \end{array}\right]$$

We eliminate the x variables from the second and fourth equations:

$$\begin{array}{c}\text{Row 2: subtract Row 1}\\ \text{Row 4: subtract } 3\times\text{Row 1}\end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3\\ 0 & 10 & 1 & -2\\ 0 & 2 & 1 & 4\\ 0 & 10 & 1 & -2 \end{array}\right]$$

We then divide Row 2 by 10 to get a 1 in the pivot position:

$$\text{Row 2: divide by 10} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3\\ 0 & 1 & 0.1 & -0.2\\ 0 & 2 & 1 & 4\\ 0 & 10 & 1 & -2 \end{array}\right]$$

Next, we eliminate the y variables from the third and fourth equations:

$$\begin{array}{c}\text{Row 3: subtract } 2\times\text{Row 2}\\ \text{Row 4: subtract } 10\times\text{Row 2}\end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3\\ 0 & 1 & 0.1 & -0.2\\ 0 & 0 & 0.8 & 4.4\\ 0 & 0 & 0 & 0 \end{array}\right]$$

We divide Row 3 by 0.8 to get a 1 in the pivot position:

$$\text{Row 3: divide by 0.8} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3\\ 0 & 1 & 0.1 & -0.2\\ 0 & 0 & 1 & 5.5\\ 0 & 0 & 0 & 0 \end{array}\right]$$

We then proceed with back-substitution:

$$\begin{array}{c} \text{Row 2: subtract } 0.1\times\text{Row 3} \\ \text{Row 1: subtract } 5\times\text{Row 3} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 0 & -24.5 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This final matrix is in reduced row echelon form. The first three rows of the matrix have non-zero elements in pivot position, for a system with three unknowns, and the fourth row is a row of zeros, so we can conclude there is a unique solution: $x = -23$, $y = -0.75$, and $z = 5.5$.

(c) Consider the following system:

$$\begin{aligned} 4x + 4y + 4z + w + v &= 1 \\ x + y + 2z + 4w + v &= 2 \\ 5x + 5y + 5z + w + v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Determine the solutions to the set of equations.

Solution:

We first note that the given augmented matrix is in reduced row echelon form, which makes sense as it is the final output of the Gaussian elimination algorithm. We observe that the second and fifth columns do not have 1s in pivot position so there are two free variables corresponding to y and v.

Let $y = s$ and let $v = t$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

Using back substitution, we can solve for x, y, z, w, and v in terms of s and t:

$$\begin{array}{lll} \text{Row 1:} & x + y + 3v = 16 & \implies x = 16 - 3t - s \\ \text{Row 2:} & z - 3v = -17 & \implies z = -17 + 3t \\ \text{Row 3:} & w + v = 5 & \implies w = 5 - t \end{array}$$

The solutions to the system of equations are therefore:

$$\begin{aligned} x &= 16 - 3t - s \\ y &= s \\ z &= -17 + 3t \\ w &= 5 - t \\ v &= t \end{aligned}$$

The solutions can also be represented by a set as:

$$S = \left\{ \vec{u} \;\middle|\; \vec{u} = \begin{bmatrix} 16 \\ 0 \\ -17 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} t \quad , \quad s \in \mathbb{R}, \quad t \in \mathbb{R} \right\}.$$

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.