EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 1

1. **Reading Assignment**

For this homework, please read Note 0, Note 1A, and Note 1B. These will provide an overview of linear equations and augmented matrices. You are always welcome and encouraged to read ahead as well. How does the content you read in these notes relate to what you've learned before? What content is unfamiliar or new?

Solution: Give yourself credit for any reasonable answer.

2. **Counting Solutions**

Learning Goal: *(This problem is designed to illustrate the different types of systems of equations. Some sub-parts will have a unique solution and others have no solutions or infinitely many solutions. In this class, we will build up the mathematical machinery to systematically determine which case applies.)*

Directions: For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. If there is a unique solution, find it. If there is an infinite number of solutions, explicitly state this, describe the set of all solutions, and then write one of such solutions. You can represent the set of all solutions by writing two variables as a function of the last variable (i.e. $x = 2z+1, y = 4z-4$). If there is no solution, explain why. **Show your work**.

Example: The below example shows how to methodically solve systems of linear equations using the substitution method.

$$\begin{aligned} 2x + 3y &= 5 \\ x + y &= 2 \end{aligned}$$

Example Solution

$$2x+3y = 5 \tag{1}$$
$$x+y = 2 \tag{2}$$

Subtract: Eq (1) - 2*Eq (2)

$$y = 1 \tag{3}$$

Now we plug in Eq (3) into Eq (2) and solve for x

$$x+1 = 2$$
$$\rightarrow \quad x = 1 \tag{4}$$

From Eq (3) and Eq (4), we get the unique solution:

$$x = 1$$
$$y = 1$$

(a)

$$\begin{aligned} x + y + z &= 3 \\ 2x + 2y + 2z &= 5 \end{aligned}$$

Solution: For each subpart, we include two solutions, one using substitution (as directed in the problem) and one using Gaussian elimination. You may give yourself credit for either approach.

Solution A:

$$x + y + z = 3 \tag{5}$$

$$2x + 2y + 2z = 5 \tag{6}$$

Subtract: (6) - 2*(5)

$$0 = -1 \tag{7}$$

We see this results in a contradiction in (7), indicating that no values of x,y,z can satisfy both equations. Therefore there are no solutions.

Solution B:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$

No solution. The fact that there are fewer equations than there are unknowns immediately means that it is not possible to have a unique solution (we cannot determine a precise value for any variable because we will not have enough equations to properly substitute out all the other variables); however, this does not guarantee that there is a solution to begin with. From Gaussian Elimination, we can see that these equations are contradictory since $0 \neq -1$. In other words, no values of x, y, and z can satisfy both equations simultaneously.

(b)

$$\begin{aligned} -y + 2z &= 1 \\ 2x + z &= 2 \end{aligned}$$

Solution:

Solution A:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

We notice that because we cannot cancel out x or y using the other equation, the equations do not contradict each other so there must exist an infinite number of solutions. We choose z to be our free variable and can then solve each equation in terms of z.

$$x = 1 - \frac{1}{2}z$$

$$y = 2z - 1$$

Solution B:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \end{array}\right] \text{swapping } R_1 \text{ and } R_2$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & -1 & 2 & 1 \end{array}\right] \text{using } R_1 \leftarrow \tfrac{1}{2}R_1$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & -1 \end{array}\right] \text{using } R_2 \leftarrow -R_2$$

We have now completed Gaussian elimination because we have a leading 1 in each row with zeros below that 1 in its column. In this way we can explicitly see that z is a free variable (x and y depend on z and there are no constraints on the value of z). Thus there are an infinite number of solutions. The set of infinite solutions has the form (for some $z \in \mathbb{R}$):

$$x = 1 - \frac{1}{2}z$$
$$y = 2z - 1$$

To get full credit it is enough to state "Infinite solutions" and give one possible solution that fits the form above.

(c)

$$\begin{array}{rcrl} x & + & 2y & = 5 \\ 2x & - & y & = 0 \\ 3x & + & y & = 5 \end{array}$$

Solution:

Solution A:

In this case, there are three equations with only two unknowns. However, this fact alone does not tell us whether there is a unique solution, no solution, or an infinite number of solutions.

$$x + 2y = 5 \tag{8}$$
$$2x - y = 0 \tag{9}$$
$$3x + y = 5 \tag{10}$$

Adding (8) and (9), we obtain

$$3x + y = 5 \tag{11}$$

Notice that equation (11) = equation (10)! Put another way, equation (10) provides no new information about the system that equations (8) and (9) could not tell us (importantly, it also does not contradict any information from the previous equations as well). Knowing this, we focus only on (8) and (9).

Add (8) + 2*(9)

$$5x = 5$$
$$\rightarrow \quad x = 1 \tag{12}$$

Plugging this value of x back into (8), we obtain

$$1 + 2y = 5$$
$$\rightarrow \quad y = 2 \tag{13}$$

Yielding the unique solution

$$x = 1$$
$$y = 2$$

Solution B:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 3 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 3 & 1 & 5 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$
$$\to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{array}\right] \text{using } R_3 \leftarrow R_3 - 3R_1$$
$$\to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & -5 & -10 \end{array}\right] \text{using } R_2 \leftarrow -\tfrac{1}{5}R_2$$
$$\to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \text{using } R_3 \leftarrow R_3 + 5R_2$$
$$\to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \text{using } R_1 \leftarrow R_1 - 2R_2$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The system of linear equations at the end of the Gaussian Elimination above simply reads out

$$x = 1$$
$$y = 2$$
$$0 = 0$$

(d)

$$\begin{array}{rcrl} x & + & 2y & = 3 \\ 2x & - & y & = 1 \\ x & - & 3y & = -5 \end{array}$$

Solution:

Solution A:

$$x + 2y = 3 \tag{14}$$
$$2x - y = 1 \tag{15}$$
$$x - 3y = -5 \tag{16}$$

Add: (14) + (16)

$$2x - y = -2 \tag{17}$$

Subtract: (15) - (17)

$$0 = 3$$

This is a contradiction, so there is no solution.

There is no solution for this system as no choice of x and y can satisfy all equations simultaneously. This is often what happens when you have more equations than unknowns, although as you saw in the previous part, it doesn't always happen.

Solution B:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 1 & -3 & -5 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -8 \end{array}\right] \text{using } R_3 \leftarrow R_3 - R_1$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -8 \end{array}\right] \text{using } R_2 \leftarrow -\tfrac{1}{5}R_2$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right] \text{using } R_3 \leftarrow R_3 + 5R_2$$

No solution. We can think of this to mean that there are no values of x and y which satisfy the conditions in all three equations simultaneously, because in order to satisfy all three equations, the last row $0 = -3$ would need to be true.

3. Magic Square

In an $n \times n$ "magic square," all of the sums across each of the n rows, n columns, and 2 diagonals equal magic constant k. For example, in the below magic square, each row, column, and diagonal sums to 34.

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

The magic square is a classic math puzzle, and some of you may have solved these as children by guessing. However, it turns out they can be solved systematically by setting up a system of linear equations!

(a) How many linear equations can you write for an $n \times n$ magic square?

Solution:

$2n + 2$, since there is one equation for each of the n rows, n columns, and 2 diagonals.

(b) For the generalized magic square below, write out a system of linear equations.
Hint: Set the sum of entries in each row, column, and diagonal equal to k.

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

Solution:

$$\begin{aligned}
x_{11}+x_{12}+x_{13}&=k\\
x_{21}+x_{22}+x_{23}&=k\\
x_{31}+x_{32}+x_{33}&=k\\
x_{11}+x_{21}+x_{31}&=k\\
x_{12}+x_{22}+x_{32}&=k\\
x_{13}+x_{23}+x_{33}&=k\\
x_{11}+x_{22}+x_{33}&=k\\
x_{31}+x_{22}+x_{13}&=k
\end{aligned}$$

(c) Now consider the following square, with some entries filled in. Substitute the known entries into the linear equations you wrote in part (b) to solve for the missing entries x_{11}, x_{12}, x_{32}. Please show the equations you use to solve; credit will not be given for solving by inspection.

x_{11}	x_{12}	8
9	5	1
2	x_{32}	6

Solution:

$$x_{11}+x_{12}+8=k \tag{18}$$
$$9+5+1=k \tag{19}$$
$$2+x_{32}+6=k \tag{20}$$
$$x_{11}+9+2=k \tag{21}$$
$$x_{12}+5+x_{32}=k \tag{22}$$
$$8+1+6=k \tag{23}$$
$$x_{11}+5+6=k \tag{24}$$
$$2+5+8=k \tag{25}$$

From Eq. 19, $k = 15$.
Substituting $k = 15$ back into Eq. 20, $x_{32} = 7$.
Similarly, substituting $k = 15$ back into Eq. 24, $x_{11} = 4$.
Finally, substituting $k = 15, x_{32} = 7$ into Eq. 22, we find $x_{12} = 3$.

4. Word Problems

Learning Objective: *Understand how to setup a system of linear equations from word problems.*

For these word problems, represent the system of linear equations as an augmented matrix. Then, solve the system using substutition or Gaussian elimination.

(a) Gustav is collecting soil samples. Each soil sample contains some sand, some clay, and some organic material. He wants to know the density of each material. His first sample has 0.5 liters of sand, 0.25 liters of clay, and 0.25 liters of organic material, and weighs 1.625 kg. His second sample contains 1 liter of sand, 0 liters of clay, and 1 liter of organic material, and weighs 3 kg. His third sample contains 0.25 liters of sand, 0.5 liters of clay, and 0 liters of organic material, and weighs 1.375 kg. That is,

$$0.5s + 0.25c + 0.25m = 1.625 \tag{26}$$
$$1s + 0c + 1m = 3 \tag{27}$$
$$0.25s + 0.5c + 0m = 1.375 \tag{28}$$

where s is the density of sand, c is the density of clay, and m is the density of organic material, all measured in kg/L. Solve for the density of each material.

Solution: We translate the system of equations to an augmented matrix and solve.

$$\left[\begin{array}{ccc|c} 0.5 & 0.25 & 0.25 & 1.625 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right]$$

To solve, we use substitution. Rearranging the second equation,

$$s = 3 - m$$

Rearranging the third equation,

$$0.5c = 1.375 - 0.25s$$

Plugging in $s = 3 - m$ into this new equation and multiplying both sides by 2, we find that

$$c = 2.75 - 0.5(3 - m)$$

Now that we have an expression for s and c in terms of m, we can plug these expressions into the first equation to solve for m. This yields

$$0.5(3 - m) + 0.25(2.75 - 0.5(3 - m)) + 0.25m = 1.625$$
$$m = 1.5$$

Backsubstituting into the second equation yields

$$s = 1.5$$

And again into the third equation

$$c = 2$$

We find sand has a density of 1.5 kg/L, clay has a density of 2 kg/L, and organic material has a density of 1.5 kg/L.

Alternatively, using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 0.5 & 0.25 & 0.25 & 1.625 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \text{using } R_1 \leftarrow 2R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 1 & 2 & 0 & 5.5 \end{array}\right] \text{using } R_3 \leftarrow 4R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0 & 1.5 & -0.5 & 2.25 \end{array}\right] \text{using } R_3 \leftarrow R_3 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_3 \leftarrow R_3 + 3R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0 & -1 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_2 \leftarrow R_2 - \tfrac{1}{2}R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_2 \leftarrow -2R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_1 \leftarrow R_1 - \tfrac{1}{2}R_2 - \tfrac{1}{2}R_3$$

(b) Alice buys 3 apples and 4 oranges for 17 dollars. Bob buys 1 apple and 10 oranges for 23 dollars (Bob really likes oranges). How much do apples and oranges cost individually?

Solution: Treat x as the cost of one apple and y as the cost of one orange.

$$\left[\begin{array}{cc|c} 3 & 4 & 17 \\ 1 & 10 & 23 \end{array}\right]$$

Using substitution, we first isolate x from the first equation.

$$x = \frac{17}{3} - \frac{4}{3}y$$

Plugging this into the second equation, we find

$$\frac{17}{3} - \frac{4}{3}y + 10y = 23$$

Solving for y yields

$$y = 2$$

Backsubstition into the first equation yields

$$3x + 4 \times 2 = 17$$

$$x = 3$$

Alternatively, using Gaussian elimination:

$$\left[\begin{array}{cc|c} 3 & 4 & 17 \\ 1 & 10 & 23 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 1 & 10 & 23 \end{array}\right] \text{using } R_1 \leftarrow \tfrac{1}{3}R_1$$
$$\to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 0 & \frac{26}{3} & \frac{52}{3} \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1$$
$$\to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 0 & 1 & 2 \end{array}\right] \text{using } R_2 \leftarrow \tfrac{3}{26}R_2$$
$$\to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \text{using } R_1 \leftarrow R_1 - \tfrac{4}{3}R_2$$

We find apples cost \$3 each and oranges cost \$2 each.

(c) Jack, Jill, and James are driving from Berkeley to Las Vegas. Each of them takes a different route. Jack takes a short route and ends up going through Toll Road A and Toll Road B, costing him \$10. Jill takes a slightly longer route and goes through Toll Road B and Toll Road C, costing her \$15. Finally, James takes a wrong turn and takes Toll Road A twice, then takes Toll Road B and finally Toll Road C, costing him \$25. What is the toll cost on each road?

Solution: Treat a, b, c as the toll cost on each of Toll Roads A, B, and C respectively.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 2 & 1 & 1 & 25 \end{array}\right]$$

Solving with substitution, we start with the second equation and solve for b

$$b = 15 - c$$

Rearranging the first equation and plugging in our previous results yields

$$a = 10 - b = 10 - (15 - c) = c - 5$$

Plugging both of these equations into the last equation, we find

$$2(c-5) + (15 - c) + c = 25$$

$$c = 10$$

Backsubstition yields

$$a = 5$$

$$b = 5$$

Alternatively, using Gaussian elimination:

$$
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 2 & 1 & 1 & 25 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & -1 & 1 & 5 \end{array}\right] \text{using } R_3 \leftarrow R_3 - 2R_1
$$

$$
\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 2 & 20 \end{array}\right] \text{using } R_3 \leftarrow R_3 + R_2
$$

$$
\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_3 \leftarrow \tfrac{1}{2}R_3
$$

$$
\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_3
$$

$$
\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_1 \leftarrow R_1 - R_2
$$

We find Toll Road A cost \$5, Toll Road B cost \$5, and Toll Road C cost \$10.

5. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Please remember to submit both your homework as well as the self-grade assignment following the release of the solutions. A full description of the submission process is listed on the class website (eecs16a.org).

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.