EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 5

This homework is due Friday, February 25, 2022 at 23:59.
Self-grades are due Monday, February 28, 2022 at 23:59.

Submission Format
Your homework submission should consist of **one** file.

1. **Reading Assignment**

   For this homework, please read Note 7, 8, and 9. These notes will give you an overview of matrix subspaces and eigenvalues/eigenvectors. You are always welcome and encouraged to read beyond this as well.

2. **Subspaces, Bases and Dimension**

   For each of the sets $\mathbb{U}$ (which are subsets of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a subspace of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

   (a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

   (b) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

   (c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

   (d) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+y^2 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

3. **Finding Null Spaces and Column Spaces**

   Learning Objectives: *Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

   Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as Null($\mathbf{A}$) and the definition can be written in set notation as:

   $$Null(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

   Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of $\mathbf{A}$'s columns. The column space can be notated as Col($\mathbf{A}$) or range($\mathbf{A}$) and the definition can be written in set notation as:

   $$Col(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors - i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of Col($\mathbf{A}$)?

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span Col($\mathbf{A}$) (i.e. a basis for Col($\mathbf{A}$)). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of Col($\mathbf{A}$)?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

(c) Find a *minimum* set of vectors that span Null($\mathbf{A}$) (i.e. a basis for Null($\mathbf{A}$)), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of Null($\mathbf{A}$)?

(d) Find the sum of the dimensions of Null($\mathbf{A}$) and Col($\mathbf{A}$). What do you notice about this sum in relation to the dimensions of $\mathbf{A}$?

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span Col($\mathbf{B}$) (i.e. a basis for Col($\mathbf{B}$)). What is the minimum number of vectors required to span the Col($\mathbf{B}$)?

(f) You are given the following matrix $\mathbf{G}$. Find a *minimum* set of vectors that span Null($\mathbf{G}$), i.e. a basis for Null($\mathbf{G}$).

$$\mathbf{G} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

(g) For the following matrix $\mathbf{D}$, find Col($\mathbf{D}$) and its dimension, and Null($\mathbf{D}$) and its dimension.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

4. Linear Dependence in a Square Matrix

Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent. You can use the following conclusion in your proof:

If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

5. Mechanical Determinants

For each of the following matrices, compute their determinant and state whether they are invertible.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

(c) $\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$.

(d) $\begin{bmatrix} -4 & 2 & 1 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

(e) $\begin{bmatrix} -4 & 0 & 0 \\ 5 & 1 & -3 \\ 7 & 3 & 1 \end{bmatrix}$.

(f) $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

6. Introduction to Eigenvalues and Eigenvectors

Learning Goal: *Practice calculating eigenvalues and eigenvectors. The importance of eigenvalues and eigenvectors will become clear in the following problems.*

For each of the following matrices, find their eigenvalues and the corresponding eigenvectors. For simple matrices, you may do this by inspection if you prefer.

(a) $\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a general square matrix. Show that the set of eigenvectors corresponding to a particular eigenvalue of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. In other words, show that

$$\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \lambda \vec{x}, \lambda \in \mathbb{R}\}$$

is a subspace. You have to show that all three properties of a subspace (as mentioned in Note 8) hold.

7. Is There A Steady State?

So far, we've seen that for a conservative state transition matrix $\mathbf{A}$, we can find the eigenvector, $\vec{v}$, corresponding to the eigenvalue $\lambda = 1$. This vector is the steady state since $\mathbf{A}\vec{v} = \vec{v}$. However, we've so far taken for granted that the state transition matrix even has the eigenvalue $\lambda = 1$. Let's try to prove this fact.

(a) Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, then it is also an eigenvalue of the matrix $\mathbf{A}^T$.
Hint: The determinants of $\mathbf{A}$ and $\mathbf{A}^T$ are the same. This is because the volumes which these matrices represent are the same.

(b) Let a square matrix $\mathbf{A}$ have, for each row, entries that sum to one. Show that $\vec{1} = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$. What is the corresponding eigenvalue?

(c) Let's put it together now. From the previous two parts, show that any conservative state transition matrix will have the eigenvalue $\lambda = 1$. Recall that conservative state transition matrices have, for each column, entries that sum to 1.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

