EECS 16A Spring 2022 — Designing Information Devices and Systems I — Discussion 13A

1. Least Squares with Orthogonal Columns

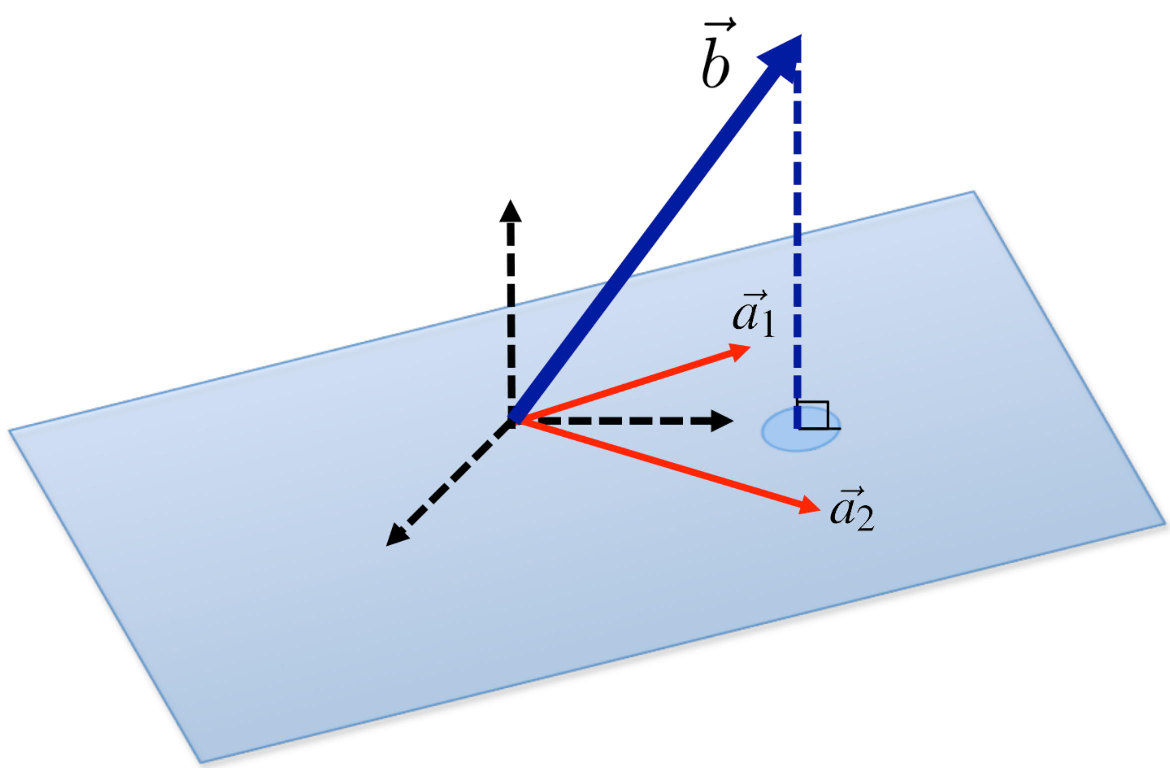
(a) Consider a least squares problem of the form

$$\min_{\vec{x}} \left\| \vec{b} - \mathbf{A}\vec{x} \right\|^2 = \min_{\vec{x}} \left\| \mathbf{A}\vec{x} - \vec{b} \right\|^2 = \min_{\vec{x}} \left\| \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2$$

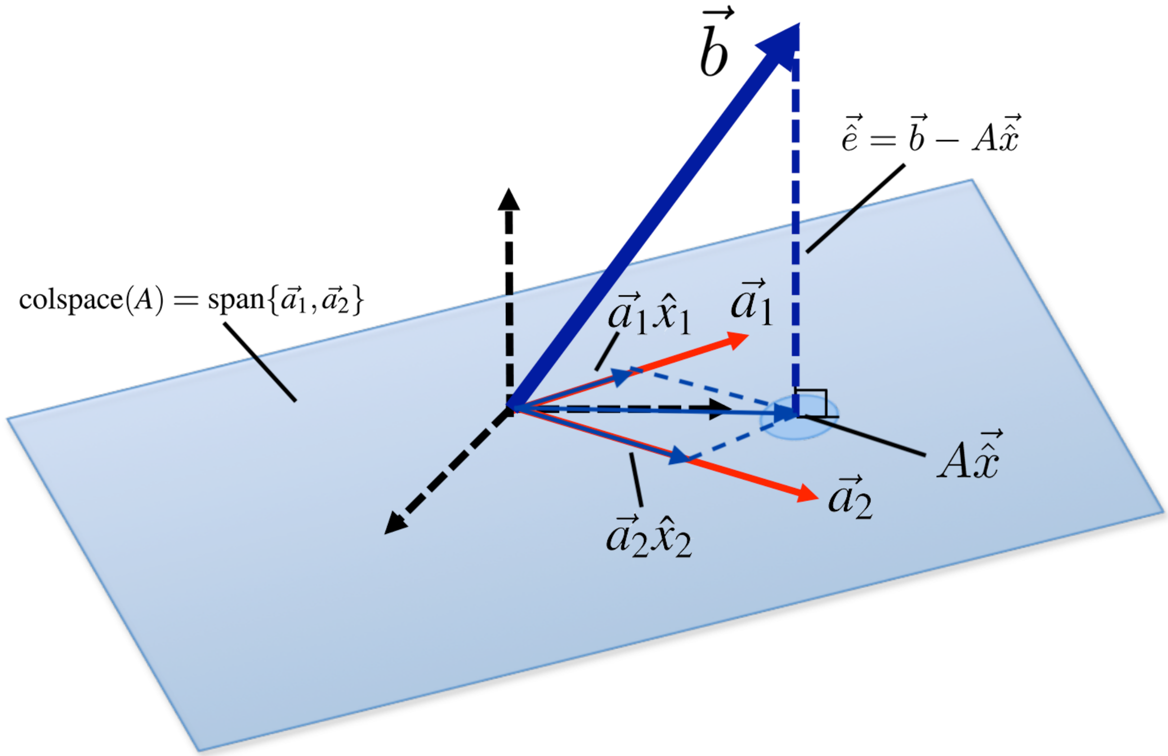
Let the solution be $\vec{\hat{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$.

Label the following elements in the diagram below.

$$\text{span}\{\vec{a}_1, \vec{a}_2\}, \qquad \vec{e} = \vec{b} - \mathbf{A}\vec{\hat{x}}, \qquad \mathbf{A}\vec{\hat{x}}, \qquad \vec{a}_1\hat{x}_1,\ \vec{a}_2\hat{x}_2, \qquad \text{colspace}(\mathbf{A})$$

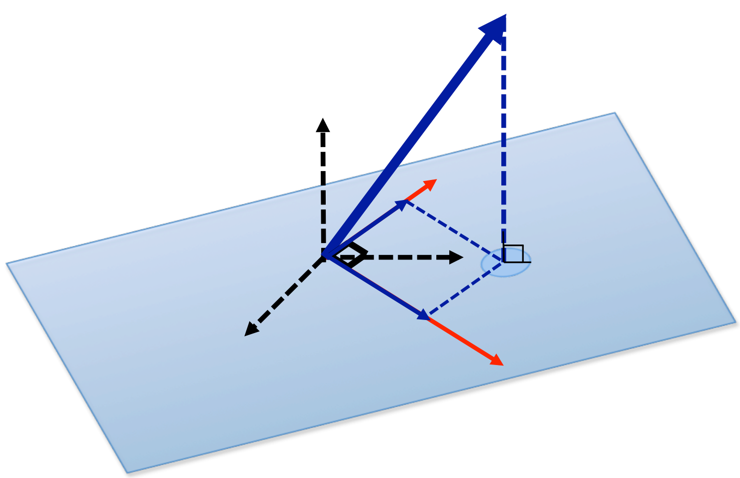


Answer:



(b) We now consider the special case of least squares where the columns of $\mathbf{A}$ are orthogonal (illustrated in the figure below). Given that $\hat{\vec{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$ and $A\hat{\vec{x}} = \text{proj}_{\mathbf{A}}(\vec{b}) = \hat{x}_1\vec{a}_1 + \hat{x}_2\vec{a}_2$, show that

$$\text{proj}_{\vec{a}_1}(\vec{b}) = \hat{x}_1\vec{a}_1$$
$$\text{proj}_{\vec{a}_2}(\vec{b}) = \hat{x}_2\vec{a}_2$$



Answer: The projection of $\vec{b}$ onto $\vec{a}_1$ and $\vec{a}_2$ are given by:

$$\text{proj}_{\vec{a}_1}(\vec{b}) = \frac{\langle \vec{a}_1, \vec{b} \rangle}{\|\vec{a}_1\|^2}\vec{a}_1 \qquad \text{proj}_{\vec{a}_2}(\vec{b}) = \frac{\langle \vec{a}_2, \vec{b} \rangle}{\|\vec{a}_2\|^2}\vec{a}_2$$

$$\text{Length:} \quad \frac{\langle \vec{a}_1, \vec{b} \rangle}{\|\vec{a}_1\|} \qquad \frac{\langle \vec{a}_2, \vec{b} \rangle}{\|\vec{a}_2\|}$$

The least squares solution is given by:

$$\begin{aligned}
\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} &= \left(\begin{bmatrix} - & \vec{a}_1{}^T & - \\ - & \vec{a}_2{}^T & - \end{bmatrix} \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \right)^{-1} \begin{bmatrix} - & \vec{a}_1{}^T & - \\ - & \vec{a}_2{}^T & - \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\|\vec{a}_1\|^2} & 0 \\ 0 & \frac{1}{\|\vec{a}_2\|^2} \end{bmatrix} \begin{bmatrix} - & \vec{a}_1{}^T & - \\ - & \vec{a}_2{}^T & - \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\vec{a}_1{}^T\vec{b}}{\|\vec{a}_1\|^2} \\ \frac{\vec{a}_2{}^T\vec{b}}{\|\vec{a}_2\|^2} \end{bmatrix}
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{proj}_{\vec{a}_1}(\vec{b}) &= \frac{\langle \vec{a}_1, \vec{b} \rangle}{\|\vec{a}_1\|^2}\vec{a}_1 = \frac{\vec{a}_1{}^T\vec{b}}{\|\vec{a}_1\|^2}\vec{a}_1 = \hat{x}_1\vec{a}_1 \\
\text{proj}_{\vec{a}_2}(\vec{b}) &= \frac{\langle \vec{a}_2, \vec{b} \rangle}{\|\vec{a}_2\|^2}\vec{a}_2 = \frac{\vec{a}_2{}^T\vec{b}}{\|\vec{a}_2\|^2}\vec{a}_2 = \hat{x}_2\vec{a}_2
\end{aligned}$$

(c) Compute the least squares solution to

$$\min_{\vec{x}} \quad \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2.$$

Answer: Using least squares again,

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Note that the columns of $\mathbf{A}$ are orthogonal, so it is much faster to project $\vec{b}$ onto the columns of $\mathbf{A}$ than use the least squares formula to find $\vec{x}$.

2. Building a Classifier

We would like to develop a classifier to classify points based on their distance from the origin.

You are presented with the following data. Each data point $\vec{d}_i^{\,T} = [x_i \; y_i]^T$ has the corresponding label $l_i \in \{-1, 1\}$.

x_i	y_i	l_i
-2	1	-1
-1	1	1
1	1	1
2	1	-1

Table 1: *
Labels for data you are classifying

(a) You want to build a model to understand the data. You first consider a linear model, i.e. you want to find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $l_i \approx \alpha x_i + \beta y_i + \gamma$.

Set up a least squares problem to solve for α, β, and γ. If this problem is solvable, solve it, i.e. find the best values for α, β, γ. If it is not solvable, justify why.

Rewriting the equations $\alpha x_i + \beta y_i + \gamma \approx l_i$ for $i = 1,2,3,4$ in matrix form gives:

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \approx \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{A}\vec{x} \approx \vec{b}$$

The least squares solution is $\hat{\vec{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$. The solution only exists when the matrix $\mathbf{A}^T\mathbf{A}$ is invertible, and an equivalent condition is when all the columns of $\mathbf{A}$ are linearly independent. We see that the second and third columns of $\mathbf{A}$ are linearly dependent, so the problem is **not** solvable.

(b) You now consider a model with a quadratic term: $l_i \approx \alpha x_i + \beta x_i{}^2$ with $\alpha, \beta \in \mathbb{R}$. *Read the equation carefully!*

Set up a least squares problem to fit the model to the data. If this problem is solvable, solve it, i.e, find the best values for α, β. If it is not solvable, justify why.

x_i	y_i	l_i
-2	1	-1
-1	1	1
1	1	1
2	1	-1

Table 2: *
Labels for data you are classifying

Rewriting the equations $\alpha x_i + \beta x_i^2 \approx l_i$ for $i = 1,2,3,4$ in matrix form gives:

$$\begin{bmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \\ x_3 & x_3^2 \\ x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \approx \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{A}\vec{x} \approx \vec{b}$$

The least squares solution is $\hat{\vec{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$. The solution only exists when the matrix $\mathbf{A}^T\mathbf{A}$ is invertible, and an equivalent condition is when all the columns of $\mathbf{A}$ are linearly independent. We see that the first and second columns of $\mathbf{A}$ are linearly independent, so the problem is solvable.

We can solve for $\hat{\vec{x}} = \left(\begin{bmatrix} -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix}\right)^{-1} \begin{bmatrix} -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \end{bmatrix}^T = \begin{bmatrix} 0 & \frac{-3}{17} \end{bmatrix}^T$.

(c) Finally, you consider the model: $l_i \approx \alpha x_i + \beta {x_i}^2 + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Independent of the work you have done so far, **would you expect this model or the model in part (b) (i.e. $l_i \approx \alpha x_i + \beta {x_i}^2$) to have a smaller error in fitting the data? Explain why.**

We expect the model in part (c) to have a smaller error because there are more degrees of freedom. The model in part (b) only considers lines passing through the origin, while the model in part (c) considers all lines. With the model in part (e) we are able to obtain a line $x_i^2 = u$ where $1 < u < 4$ that would separate the data points based on their labels, unlike the model in part (b).