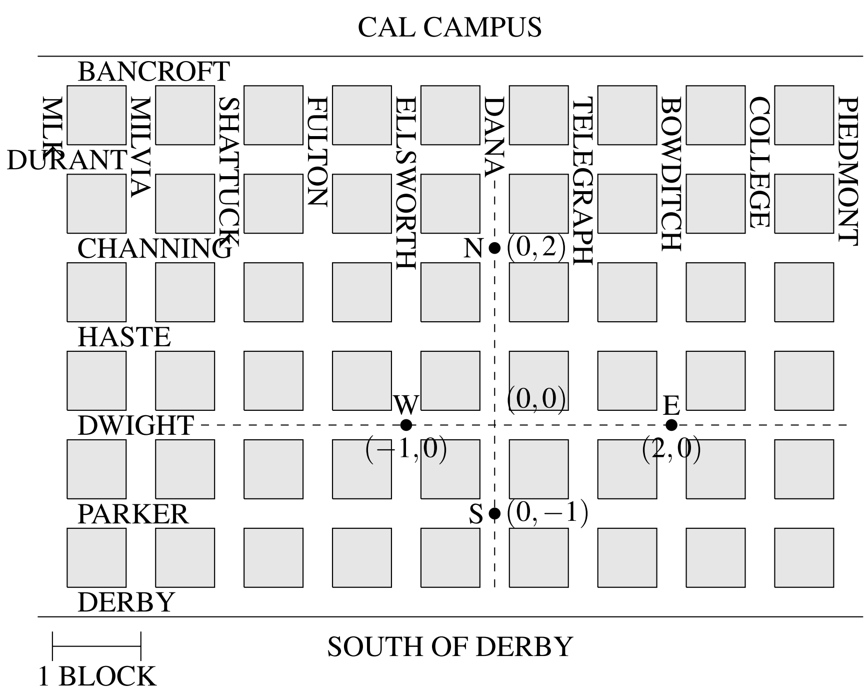
EECS 16A Designing Information Devices and Systems I
Spring 2022 Discussion 12B

1. Search and Rescue Dogs

Berkeley's Puppy Shelter needs your help! While Mr. Mochi Muffin was being walked, the volunteer let go of his leash and he is now running wild in the streets of Berkeley (which are quite dangerous)! Thankfully, all of the puppies at the shelter have a collar that sends a Bluetooth signal to receiver towers, which are spread throughout the streets (pictured below). If the puppy/collar is within range of the receiver tower, the collar will send the tower a message: the distance of the collar to the tower. Each cell tower has a range of five city blocks. Can you help the shelter locate their lost puppy?

Note: A city block is defined as the middle of an intersection to the middle of an adjacent intersection (scale provided on map). Mr. Mochi Muffin is constrained to running wild in the streets, meaning he won't be found in any buildings. If your TA asks 'Where is Mr. Mochi Muffin?' it is sufficient to answer with his intersection or 'between these two intersections.'

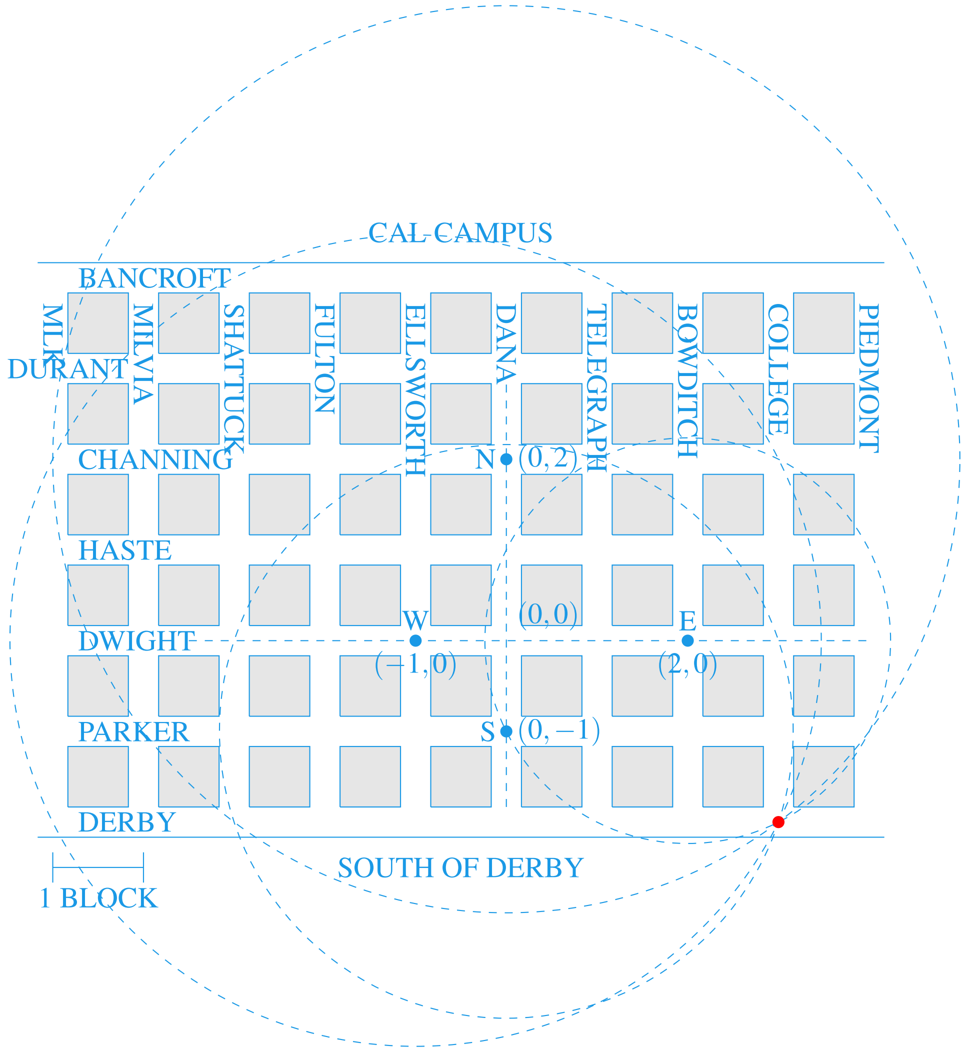


(a) You check the logs of the cell towers, and they have received the following messages:

Sensor	Distance
N	5
W	$\sqrt{20}$
E	$\sqrt{5}$
S	$\sqrt{10}$

On the map provided above, identify where Mr. Mochi Muffin is!

Answer:



(b) Can you set this up as a system of equations? Are these equations linear? If not, can these equations be linearized? If you can linearize these equations, **write down a simplified form of your set of equations.**

Hint: Set $(0,0)$ to be Dwight and Dana.

Hint 2: Distance $= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Hint 3: You don't need all 4 equations. You have two unknowns, x and y. You know from lecture that you need at least three circles to uniquely find a point on a 2D plane. How can you use the third circle/equation to get two equations and two unknowns?

Note: Remember to check for consistency for all nonlinear equations after finding the coordinates.

Answer: First, set up the system of equations:

$$(x-0)^2+(y-2)^2 = 5^2$$
$$(x+1)^2+(y-0)^2 = \sqrt{20}^2$$
$$(x-2)^2+(y-0)^2 = \sqrt{5}^2$$

Simplify by expanding the squared terms:

$$x^2 + y^2 - 4y + 4 = 25$$
$$x^2 + 2x + 1 + y^2 = 20$$
$$x^2 - 4x + 4 + y^2 = 5$$

Then subtract equation (1) from equations (2) and (3) to get linearized equations:

$$2x + 4y - 3 = 20 - 25$$
$$-4x + 4y = 5 - 25$$

This solves to $x = 3, y = -2$, which is roughly College and Derby.

Note that we only needed three out of the four equations to find the location of Mr. Mochi Muffin and could omit any one of the four. We can verify that this solution is valid with our fourth equation:

$$(x-0)^2 + (y+1)^2 = \sqrt{10}^2$$
$$(3-0)^2 + (-2+1)^2 = 10$$

Answer:

The above was done in scalar form. You can also do it in vector form, which is easier to generalize to higher dimensions. It's unlikely a student will do it in vector form, but you might want to show it to your students. We have below Sensor = North, East, South, West:

$$d = \sqrt{\| \text{Puppy} - \text{Sensor} \|^2}$$

$$\| \text{Puppy} \|^2 + \| \text{North} \|^2 - 2\langle \text{Puppy, North} \rangle = 5^2$$
$$\| \text{Puppy} \|^2 + \| \text{West} \|^2 - 2\langle \text{Puppy, West} \rangle = \sqrt{20}^2$$
$$\| \text{Puppy} \|^2 + \| \text{East} \|^2 - 2\langle \text{Puppy, East} \rangle = \sqrt{5}^2$$

Then do the same thing (subtract (1) from (2) and (3)) and solve!

(c) Suppose Mr. Mochi Muffin is moving fast, and by the time you get to destination in part (a) he's already run off! You check the logs of the cell towers again, and see the following updated messages:

Sensor	Distance
N	5
W	$\sqrt{20}$
E	Out of Range
S	Out of Range

Can you find Mr. Mochi Muffin? With a system of linear equations? Other methods? If so, on the map provided above, identify where Mr. Mochi Muffin is!

Answer: Once again, we can set up a system of equations, but we only have two known values for the distances, so we have information for two equations:

$$(x-0)^2 + (y-2)^2 = 5^2$$
$$(x+1)^2 + (y-0)^2 = \sqrt{20}^2$$

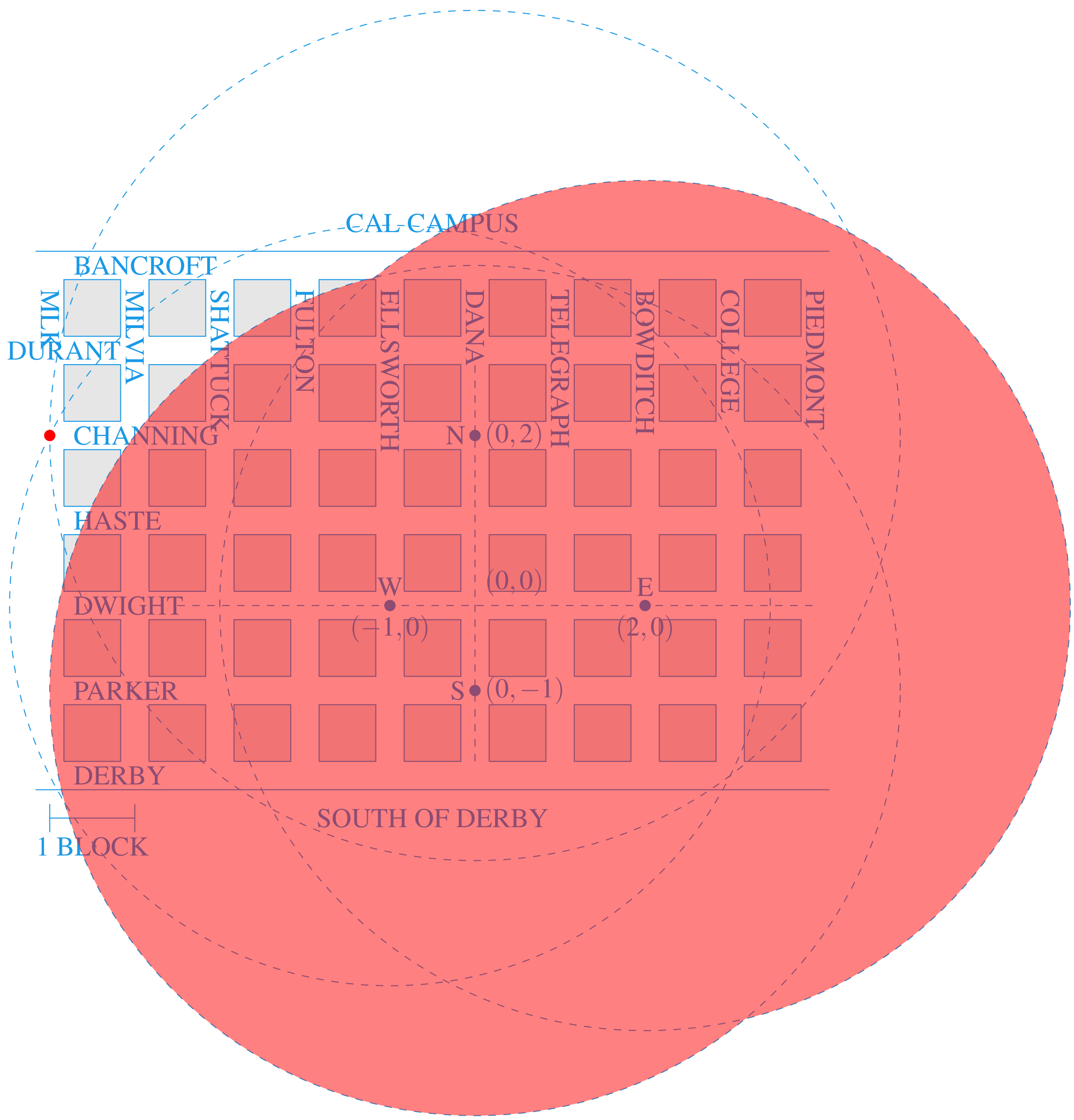
We can expand the squared terms:

$$x^2 + y^2 - 4y + 4 = 25$$
$$x^2 + 2x + 1 + y^2 = 20$$

To get linear equations, we would have to subtract one equation from the other, leaving us with just one linear equation and two unknowns, which would not give us one unique solution for where Mr. Mochi Muffin is:

$$2x + 4y - 3 = 20 - 25 = -5$$

Thus, in this question, by linearizing our system, we are losing information that could have helped us solve the system. Let's go back to a non-linear approach, which is drawing circles. With the information we have from the two sensors in range, we can draw two circles, which can find at most two possible solutions. With two out-of-range sensors, it might seem like we will not be able to find a unique solution (since we need three circles to intersect at a point). The trick is that out-of-range sensors still provide information on where Mr. Mochi Muffin is NOT located. See the diagram below - Mr. Mochi Muffin cannot be in the shaded region. Therefore, Mr. Mochi Muffin should be in Channing and MLK.



2. Mechanical Projection

In $\mathbb{R}^n$, the vector valued projection of vector $\vec{b}$ onto vector $\vec{a}$ is defined as:

$$\text{proj}_{\vec{a}}\left(\vec{b}\right) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\|\vec{a}\|^2}\vec{a}.$$

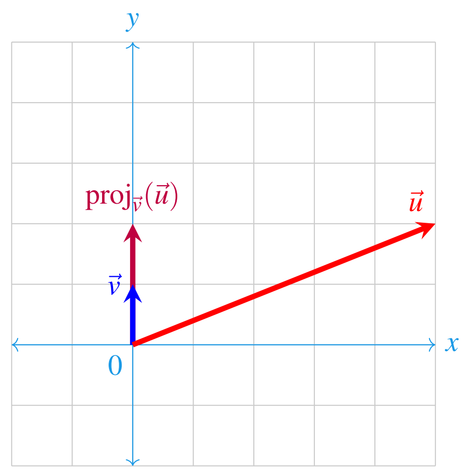
Recall $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle$.

(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ – that is, onto the y-axis. Graph these two vectors and the projection.

Answer:

$$\vec{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_{\vec{v}}(\vec{u}) &= \frac{\vec{u}^\top \vec{v}}{\|\vec{v}\|^2} \vec{v} \\ &= \frac{2}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

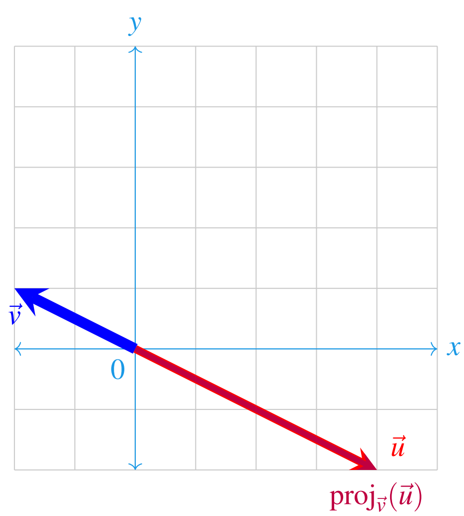


(b) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.

Answer:

$$\vec{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_{\vec{v}}(\vec{u}) &= \frac{\vec{u}^\top \vec{v}}{\|\vec{v}\|^2} \vec{v} \\ &= \frac{-10}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \end{aligned}$$



(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Graph these two vectors and the projection.

Answer:

$$\vec{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_{\vec{v}}(\vec{u}) &= \frac{\vec{u}^\top \vec{v}}{\|\vec{v}\|^2}\vec{v} \\ &= \frac{0}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$