EECS 16A Designing Information Devices and Systems I
Spring 2022 Discussion 3B

1. Proofs

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is **linearly dependent** if there exists constants c_1, c_2, … c_n such that $\sum_{i=1}^{i=n} c_i \vec{v}_i = \vec{0}$ and at least one c_i is nonzero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

(a) Suppose for some nonzero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Begin by defining column vectors $\vec{a}_1 \dots \vec{a}_n$.

$$\mathbf{A} = \begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix}$$

Thus, we can represent the multiplication $\mathbf{A}\vec{x}$ as

$$\begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \sum x_i \vec{a}_i = \vec{0}$$

Note that the equation above is the definition of linear dependence. That is, there exist coefficients, at least one which is nonzero, such that the sum of the vectors weighted by the coefficients is zero. These coefficients are the elements of the nonzero vector $\vec{x}$.

(b) For $\mathbf{A} \in \mathbb{R}^{m\times n}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Let us consider the difference of the two equations:

$$\mathbf{A}\vec{x}_1 - \mathbf{A}\vec{x}_2 = \mathbf{A}(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$

Once again, we've reached the definition of linear dependence since $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$. We can apply the results from part (a), setting $\vec{x} = \vec{x}_1 - \vec{x}_2$.

(c) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be a matrix for which there exists a nonzero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some nonzero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

Answer: The key insight is to use the linearity of matrix-vector multiplication.

By assumption, let $\vec{x}_1 \in \mathbb{R}^n$ be a solution to $\mathbf{A}\vec{x} = \vec{b}$. Then, for any $c \in \mathbb{R}$

$$\mathbf{A}(\vec{x}_1 + c\vec{y}) = \mathbf{A}\vec{x}_1 + \mathbf{A}(c\vec{y}) = \mathbf{A}\vec{x}_1 + c\mathbf{A}\vec{y} = \mathbf{A}\vec{x}_1 + \vec{0} = \mathbf{A}\vec{x}_1 = \vec{b}$$

where the first two equalities follow by linearity and the last two equalities follow from the assumptions that $\mathbf{A}\vec{y} = \vec{0}$ and that $\vec{x}_1$ is a solution to the system.

Hence, $\mathbf{A}(\vec{x}_1 + c\vec{y}) = \vec{b}$, implying that $(\vec{x}_1 + c\vec{y})$ is also a solution to $\mathbf{A}\vec{x} = \vec{b}$ for **any** constant c. Therefore, there are infinitely many solutions.

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2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices! Note that in this exercise we are applying a matrix transformation on each of the vertices of the unit square separately.

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by $15°$ and $30°$, respectively. Suggest some methods to rotate the unit square by $45°$ using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by $60°$? Your TA will show you the result in the iPython notebook.

Answer:

Apply $\mathbf{T}_1$ and $\mathbf{T}_2$ in succession to rotate the unit square by $45°$. To rotate the square by $60°$, you can either apply $\mathbf{T}_2$ twice, or if you prefer variety, apply $\mathbf{T}_1$ twice and $\mathbf{T}_2$ once.

(b) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by $60°$. Your TA will show you the result in the iPython notebook.

Answer: This matrix will look like the rotation matrix that rotates a vector by $60°$. This matrix can be composed by multiplying $\mathbf{T}_1$ by $\mathbf{T}_1$ by $\mathbf{T}_2$ (or equivalently, $\mathbf{T}_2$ by $\mathbf{T}_2$).

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this consider rotating the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \dots} = 1$.)

(*Hint: Use your trigonometric identities!*)

Answer:

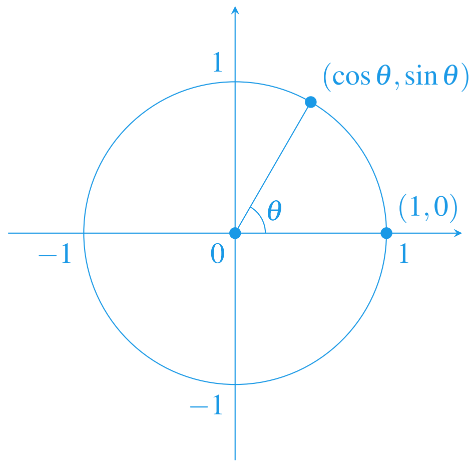
The reason the matrix is called a rotation matrix is because it transforms the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ to give $\begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}$.

Proof:

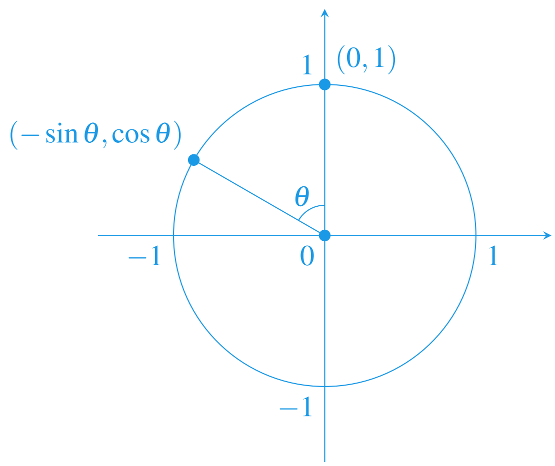
$$\begin{aligned}
\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} &= \cos\alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \sin\alpha \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \cos\alpha\sin\theta + \sin\alpha\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}
\end{aligned}$$

Alternative solution:

Let's try to derive this matrix using trigonometry. Suppose we want to rotate the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by θ.



We can use basic trigonometric relationships to see that $\begin{bmatrix}1\\0\end{bmatrix}$ rotated by θ becomes $\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}$. Similarly, rotating the vector $\begin{bmatrix}0\\1\end{bmatrix}$ by θ becomes $\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}$:



We can also scale these pre-rotated vectors to any length we want, $\begin{bmatrix}x\\0\end{bmatrix}$ and $\begin{bmatrix}0\\y\end{bmatrix}$, and we can observe graphically that they rotate to $\begin{bmatrix}x\cos\theta\\x\sin\theta\end{bmatrix}$ and $\begin{bmatrix}-y\sin\theta\\y\cos\theta\end{bmatrix}$, respectively. Rotating a vector solely in the x-direction produces a vector with both x and y components, and, likewise, rotating a vector solely in the y-direction produces a vector with both x and y components.

Finally, if we want to rotate an arbitrary vector $\begin{bmatrix}x\\y\end{bmatrix}$, we can combine what we derived above. Let x' and y' be the x and y components after rotation. x' has contributions from both x and y: $x' = x\cos\theta - y\sin\theta$. Similarly, y' has contributions from both components as well: $y' = x\sin\theta + y\cos\theta$. Expressing this in matrix form:

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta\\x\sin\theta + y\cos\theta\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Thus, we've derived the 2-dimensional rotation matrix.

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

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Answer:
Use a rotation matrix that rotates by $-60°$.

$$\begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix}$$

(e) Use part (d) to obtain the rotation matrix that reverses the operation of a matrix that rotates a vector by θ. Multiply the reverse rotation matrix with the rotation matrix and vice-versa. What do you get?
Answer:
The reverse matrix is as follows:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

We can see that for any $\vec{v} \in \mathbb{R}^2$ that the product of the rotation matrix with $\vec{v}$ followed by the product of the reverse results in the original $\vec{v}$.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{v} \right) = \vec{v}$$

(f) (For practice) Next we will look at reflection matrices. The matrix that reflects about the y axis is:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

What are the matrices that reflect a vector about the (i) x-axis, and (ii) $x = y$
Answer: The matrix that reflects about the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the matrix that reflects about $x = y$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A natural question to ask is the following: Does the *order* in which you apply these operations matter?

(g) Let's see what happens to a vector when we rotate it by $60°$ and then reflect it along the y-axis (matrix given in part (f)). Next, let's see what happens when we first reflect the vector along the y-axis and then rotate it by $60°$. You will need to multiply the corresponding rotation and reflection matrices in the correct order. Are the results the same?
Answer: The results are not the same. If you rotate some vector $\vec{v}$ and then reflect along the y-axis you get:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\cos(60°) & \sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

If you reflect along the y-axis and then rotate you get:

$$\begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\cos(60°) & -\sin(60°) \\ -\sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(h) Now lets perform the operations in part (g) on the unit square in our iPython notebook. Are the results the same?
Answer: (The results are not the same as shown in the iPython notebook.