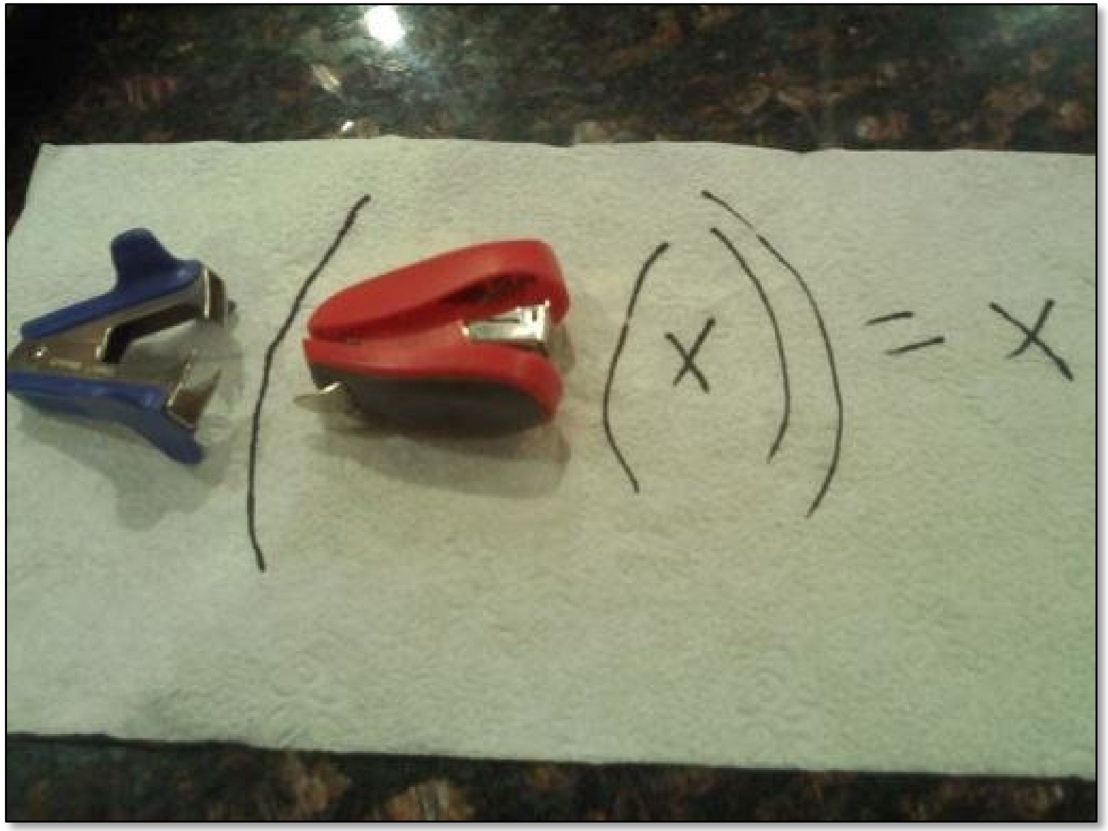


EE16A

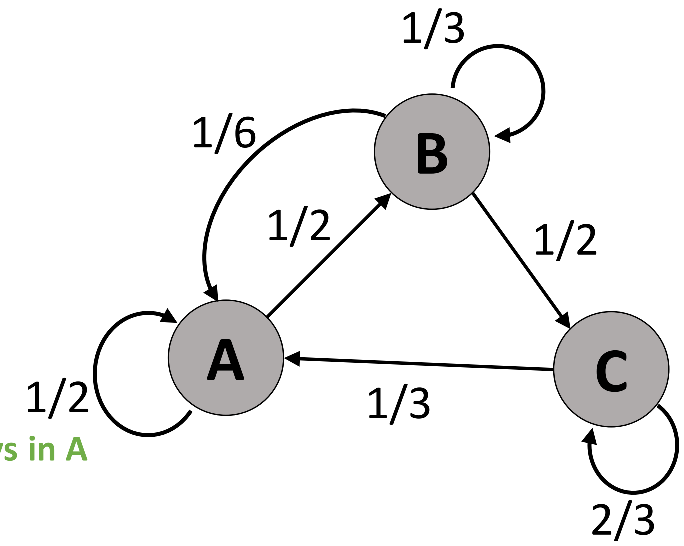
Inversion

Invertibility brings justice!

Images released by Interpol in 2007 show the 'unswirling' of the internet pictures that led to the capture of Christopher Paul Neil.

Last time: Graph Representation

Reservoirs and Pumps Example



half the water in A stays in A

Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

Where the water moves and what fraction is represented by arrows.

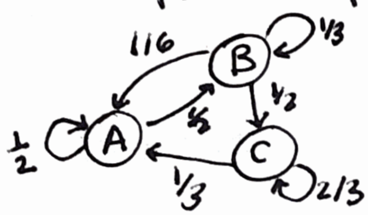
Edge weights **Edges**

"directed" graph because arrows have a direction

Note, all this stuff happens at one single time step!

Recap Last Class:

Pumps-example:



say initial water values @ $t=0$ are $\vec{x}(t=0) = \begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}$ ← water in reservoir A

level

And i run pumps once, so $t=0 \rightarrow t=1$

$$\vec{x}(t=1) = \underbrace{\begin{array}{c} \begin{array}{ccc} A\to A & B\to A & C\to A \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \overset{A\to B}{\frac{1}{2}} & \frac{1}{3} & 0 \\ \overset{A\to C}{0} & \frac{1}{2} & \frac{2}{3} \end{bmatrix} \end{array}}_{P} \underbrace{\begin{bmatrix} x_A \\ x_B \\ x_C \end{bmatrix}}_{\vec{x}(t=0)}$$

inflow for A

outflow for A

So cols represent <u>outflows</u>, rows represent <u>inflows</u>.

Does outflow have to be equal to inflow? to 1?

No, but then water is not conserved! (there is a <u>source</u> or <u>sink</u>)

What happens if i swap direction of arrows?

P gets transposed, NOT the <u>inverse</u>.

Recall, we can think of P matrix as a <u>transformation</u> of the state.
(operator)

$\boxed{\text{What if i run pumps twice?}}$

Take output of first run and use as input for second run.

<u>Ex.</u> $\vec{x}(t=0) = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \xrightarrow{\text{first run}} \vec{x}(t=1) = P \cdot \vec{x}(t=0) = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

$\xrightarrow{\text{second run}} \vec{x}(t=2) = P \cdot \vec{x}(t=1) = P \cdot P \cdot \vec{x}(t=0) = P \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2 \\ 3.5 \end{bmatrix}$

Is water conserved? Yes since sum of P col values = 1 for all cols

$\boxed{\text{What if i run pumps a ~~bajillion~~ infinite times?}}$ Is there an equilibrium state? "steady state"

i.e. does the water levels settle to a steady value?

such that in steady state, $\vec{x}_*$, we have $\vec{x}_* \xrightarrow{P} \vec{x}_*$

Written in matrix-vector multiplication form:

$$\underbrace{\vec{x}^*}_{\substack{\text{steady}\\ \text{state/equilibrium}\\ \text{water values}}} = P\vec{x}^*$$

If it exists, then equilib. input = output

Rearrange: $P\vec{x}^* - \vec{x}^* = \vec{0}$

$P\vec{x}^* - I\vec{x}^* = \vec{0}$

doesn't change anything cause $I\vec{x}^* = \vec{x}^*$ but matches up dimensions.

$$\underbrace{(P - I)}_{A}\underbrace{\vec{x}^*}_{\vec{x}} = \underbrace{\vec{0}}_{\vec{b}} \quad \text{form!}$$

$$\left(\underbrace{\begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix}}_{P \text{ From example}} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix form

$$\left[\begin{array}{ccc|c} -1/2 & 1/6 & 1/3 & 0 \\ 1/2 & -2/3 & 0 & 0 \\ 0 & 1/2 & -1/3 & 0 \end{array}\right] \xrightarrow{\text{do G.E.}}$$

system has infinite sol'ns

$$\vec{x}^* = \begin{bmatrix} 8\alpha \\ 6\alpha \\ 9\alpha \end{bmatrix}$$ for any scalar $\alpha \in \mathbb{R}$

Let's pick $\alpha = 1$

$$\boxed{\vec{x}^* = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}}$$ steady state Solution

How can we check if it's correct?

Plug into:

$$P \cdot \vec{x}^* = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 4+1+3 \\ 4+2+0 \\ 0+3+6 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}$$

output is same as input! ✓

$\boxed{\text{Recap:}}$ to find $\vec{x}$ at later time, apply P successevly.

Eg. $\vec{x}(t+1) = P \cdot \vec{x}(t)$ — one time step later; t denotes $t=0$

$\vec{x}(t+2) = P^2 \cdot \vec{x}(t)$

$\vec{x}(t+\infty) = P^{\infty} \cdot \vec{x}(t)$?

What if i want to know the water levels at a previous time?

what is $\underbrace{\vec{x}(t=-1)}_{\text{can write as } \vec{x}(t-1)}$? Given I know $\underbrace{\vec{x}(t=0)}_{\vec{x}(t)}$.

The linear transformation that describes this is called the $\boxed{\text{Inverse}}$ denoted $\cancel{A^{-1}}$ or P^{-1}

$A^{-1}\vec{x}(t) = \underbrace{\cancel{A}^{-1}P}_{I}\,\vec{x}(t-1)$ multiply both sides on Left by A^{-1}
└ not commutative

$P^{-1}\vec{x}(t) = \vec{x}(t-1)$
└ the inverse of P 'undoes' what P did

$\boxed{\text{Is it same as turning arrows backward?}}$ No! see discussion sec.

__Examples:__ What is the inverse of $f(x) = 2x$? $g(x) = \frac{1}{2}x$, ← inverse of f

so $f(g(x)) = x$

Is $f(x)=0$ invertible? No

Is eating a sandwich invertible? No (+ really)

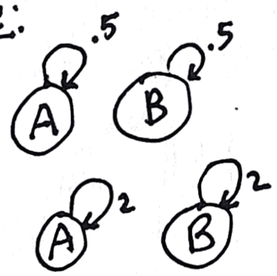
Is a scribble with iPad stylus invertible? yes, with 'undo'

Basically, invertible means we can 'undo' function & recover input.
(think about tomography problem application)

__Example:__

System R:

System Q:



$R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$

$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Let's compute:

$\vec{x}(t+1) = R \cdot \vec{x}(t)$ ← run R system

$\vec{x}(t+2) = Q \cdot \vec{x}(t+1)$ ← then Q

$= Q \cdot R\, \vec{x}(t)$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\vec{x}(t)$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}(t)$

└ nothing changes! back to input...

so $\boxed{Q \cdot R = R \cdot Q = I}$
and Q & R are inverses of each other

take 16B to know what to do if not square

Definition of Inverse matrix: Let P, Q be square matrices

P is the inverse of Q, and vice versa, if $P \cdot Q = Q \cdot P = I$

↳ we say $P = \underbrace{Q^{-1}}_{\text{"Q inverse"}}$

mtx mult. is generally NOT commutative, but inverses are

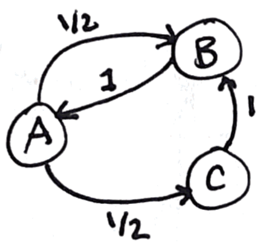
Do commutative matrices imply inverses? No!

$\boxed{\text{Properties:}}$ → Inverse is unique

→ any inverse is inverse on both left and right

→ inverse exist implies one unique sol'n to system

Example:



What is P matrix?

Write out equations:

$$\left.\begin{aligned} x_A(t+1) &= x_B(t) \\ x_B(t+1) &= \tfrac{1}{2} x_A(t) + x_C(t) \\ x_C(t+1) &= \tfrac{1}{2} x_A \end{aligned}\right\} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{How to find inverse?}}$

Want P^{-1} such that

$$\left.\begin{aligned} \vec{x}(t) &= P^{-1}\vec{x}(t+1) \\ \vec{x}(t+1) &= P\vec{x}(t) \end{aligned}\right\} \text{ need } P \cdot P^{-1} = I$$

We don't know how to divide matrices, but can solve $Ax = b$ style

$$P \cdot P^{-1} = I$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}}_{\text{unknowns to solve for}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

columns $\vec{P}_1 \quad \vec{P}_2 \quad \vec{P}_3$

$\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$

But it's mtx-mtx mult., not mtx-vector like $A\vec{x} = \vec{b}$?

treat each col as separate mtx-vector problem!

Could solve 3 G.E. $Ax = b$ style problems for $\vec{P}_1, \vec{P}_2, \vec{P}_3$ then put into matrix

BUT steps of G.E. only depend on P so can do all at once! yay! ☺

Augmented Matrix form

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1, R_2]{\text{swap}} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3-R_1]{R1 \times 2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & +1 & 0 & +1 & -1 \end{array}\right] \xrightarrow{R1-2R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

identity $\qquad P^{-1}$

Theorem: If the cols of mtx A are lin. dep.,

then A is not invertible (A^{-1} doesn't exist)

$$p \Rightarrow q \Big\} \text{ not } q \Rightarrow \text{not } p$$

if A invertible $\Rightarrow$ then cols of A are lin ~~dep~~ indep.

Proof:

Known/Beginning

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

cols lin. dep

so can write, by defn of lin. dep., there exists (Note 3)

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots c_n \vec{a}_n = \vec{0}$$

and not all c_i's are zero

need to connect cols to A^{-1}? how?

If we let A^{-1} exist:

$$A^{-1} \cdot A = AA^{-1} = I$$

can we write col or mtx in terms of other?

Write col stuff as mtx:

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

$$A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

Now what?

mult by A^{-1}! on both sides on LEFT ← not commutative

$$A^{-1}\left(A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right) = \cancel{A^{-1}} \vec{0}$$

to show:

A^{-1} does not exist

(eg. ~~orange~~ pink tigers don't exist)?

Pretend pink tigers / A^{-1} do exist, & show it generates contradiction.

"Proof by contradiction"

$$I \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

says all 0 but we said they're not

← contradiction

☺ yay! QED

Finding a matrix inverse by $\boxed{\text{Gauss-Jordan Method}}$

→ similar process as Gauss. Elim.

→ work to get into reduced Row Echelon form (RREF)

If A is invertible, want to find $B = A^{-1}$ such that $A \cdot B = I$

solve for B

~~A·x=~~

Augmented matrix form $\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{c|c} \underbrace{I}_{\text{at end of G.E.}} & A^{-1} \end{array}\right]$

What if G.E. doesn't work? Then there is no Inverse (or you made a mistake!)

Ex: $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

assume ~~a is positive here~~. $a \neq 0$ here

$aR2 - cR1 \rightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]$

divide by $(ad-bc)$ $\rightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

can't be zero or inverse doesn't exist

row echelon form, but we need RREF

$R1 \rightarrow R1 - bR2$

$\frac{ad-\cancel{bc}+\cancel{bc}}{ad-bc}$

$\left[\begin{array}{cc|cc} a & 0 & 1+\frac{bc}{ad-bc} & \frac{-ba}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

$R1/a \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

$\underbrace{}_{I} \quad \underbrace{\phantom{\frac{d}{ad-bc}}}_{A^{-1}}$

$\xrightarrow[\text{out scalar}]{\text{pull}}$ $\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

Formula for 2x2 inverse