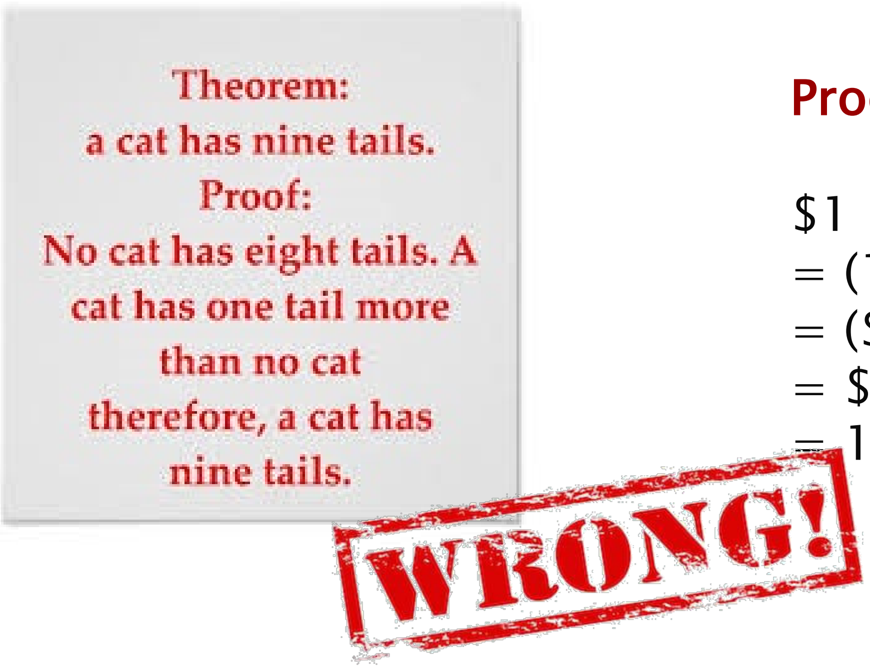

Proof that \$1 = 1 cent

\$1 = 100 cents
$= (10 \text{ cents})^2$
$= (\$0.1)^2$
= \$0.01
= 1c

EE16A

Introduction to Proofs, Span, Linear Dependence and Independence

Admin

- Warning: if you don’t have Python experience, the lab/bootcamp will be long and hard!
  - Allocate the time, work together, don’t get too discouraged

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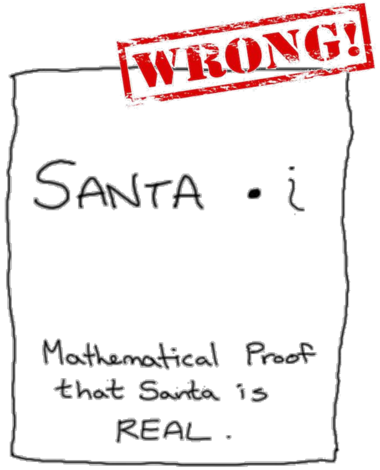
- Today:
  - Span
  - Proofs!
  - Linear (in)dependence



Last time: Multiplying matrices/vectors

$$\underset{m \times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}} = \underset{m \times p}{\begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}}$$

Must be same!

Systems of equations $A\vec{x} = \vec{b}$

$$
\begin{array}{cccc}
\text{'system' matrix} & \text{solve for me!} & & \text{measurements} \\
\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}
& \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
& =
& \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\
m \times n & n \times 1 & & m \times 1
\end{array}
$$

equations/ measurements

unknowns

Last time: Row view

Rows represent how much the variables affect a particular measurement.

$$
\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
\Rightarrow
$$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \ddots \qquad &\vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n &= b_m
\end{aligned}
$$

Last time: Column view

Columns represent how much a particular variable affects all measurements.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underbrace{x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + x_n \vec{a}_n}_{\text{Linear Combination of } \vec{a} \text{ vectors weighted by the unknowns!}} = \vec{b}$$

Linear combinations

Any **scaling or addition of vectors**

Definition:

For a set of vectors $\left\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_M\right\} \in \mathbb{R}^N$ where $\left\{\alpha_1, \alpha_2, \ldots \alpha_M\right\} \in \mathbb{R}$

then $\vec{w} \triangleq \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \ldots + \alpha_M \vec{a}_M$ is called a **linear combination** of the a-vectors

Linear combinations

<u>Any</u> **scaling or addition of vectors**

Definition:

indicates a set   of vectors   where all elements are real   each vector has dimension N

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some coefficients   are scalars

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some new vector

Example: what are some linear combinations of $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$?

Linear combinations

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Example: what are some linear combinations of $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$?

ex: $5\vec{x}_1 + 2\vec{x}_2$

$-6\vec{x}_1 - \vec{x}_2$

$0\vec{x}_1 + 5\vec{x}_2$

What does it mean to solve $A\vec{x}=\vec{b}$ w.r.t. lin. combos? ← linear combinations

with respect to

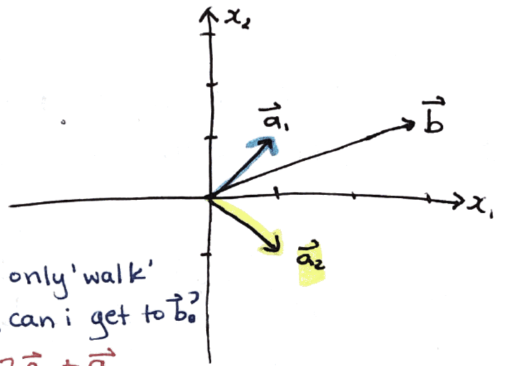
Before: find $x_1, x_2 \ldots x_n$ that satisfy eqns (rows)

Now: what lin. combo. of cols of A give $\vec{b}$? column view!

Example: $A\vec{x}=\vec{b}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



say i can only 'walk' along $\vec{a}_1, \vec{a}_2$, can i get to $\vec{b}$?

yes! with $2\vec{a}_1 + \vec{a}_2$

Try G.E. instead:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \begin{array}{l} \rightarrow x_1 = 2 \\ \rightarrow x_2 = 1 \end{array}$$

✓ yay!

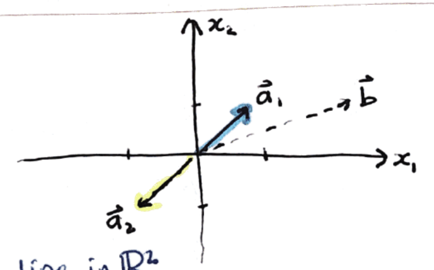
Can i reach any other $\vec{b}$ vector i want with combos of $\vec{a}_1, \vec{a}_2$?

(Note: can go backwards)

Yes! And we'll prove it later. ← as long as in $\mathbb{R}^2$

Example: $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$



Now can i reach any $\vec{b}$ vector?

No, not unless along a particular line in $\mathbb{R}^2$

$\boxed{\text{SPAN}}$ ~~of~~ the cols of A is the set of all vectors $\vec{b}$ such that $A\vec{x}=\vec{b}$ has a solution (no need to be unique)

jargon alert!

aka "Range"

"Col space of A"

i.e. the set of ~~possible~~ vectors that can be reached by all possible lin. combo. of cols of A

ample: What is the span of $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$?

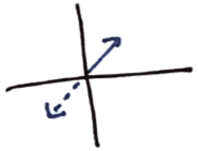
↳ entire 2D plane! Because, with $A\vec{\alpha} = \vec{b}$ we can choose α_1, α_2 such that we reach any point in $\mathbb{R}^2$

↳ switched to $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

set notation: $\text{span}(A) = \left\{ \vec{v} \mid \vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \alpha_1, \alpha_2 \in \mathbb{R} \right\}$

vectors — such that — scalars — real

What is span of $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$?



↳ A line: cannot choose α_1, α_2 to escape line

Why not? because $\vec{a}_1, \vec{a}_2$ are linearly dependent

---

Example: What is span of $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

The entire 3D space $\mathbb{R}^3$!

Example: What is the span of $\vec{0}$? $\vec{0}$ This is the one vector you can always reach (always in span)

Fancy jargon

If $\exists\ \vec{x}$ s.t. $A\vec{x} = \vec{b}$ then we say $\vec{b} \in \underbrace{\text{span}(\text{cols of } A)}_{\text{all the places you can get to}}$

there exists — a vector — such that — are in

What if $\vec{b}$ is NOT in span(cols of A)?

↳ if meas. $\vec{b}$ are not reachable by system A, then they are wrong (inconsistent) → No solution!

Let's go back to $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ case.

↳ we had 2 cols, but could only 'reach' (span) a 1D line, because cols were linearly dependent

i.e. $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\boxed{\vec{a}_2 = -\vec{a}_1}$ ~~one to~~

no new info!
Redundant redundant

$\boxed{\text{Linear Dependence}}$ of a set of vectors means at least one is redundant redundant (useless)

depends only on A

↳ let's us check whether system will work (has a unique sol'n) before we take measurements (doesn't need $\vec{b}$)

Definition 1: A set of vectors $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_M\}$ is linearly dependent if $\vec{a}_j = \sum_{m \neq j} \alpha_m \vec{a}_m$ for some $\{\alpha_m \in \mathbb{R}\}_{m \neq j}$ and some $1 \leq j \leq M$

i.e. one vector can be written as a linear combo of the others

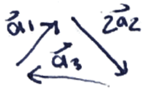
i.e. one vector is in the span of the others.

Example: $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\vec{a}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ Can any vector in set be written as lin. combo. of others?

yes! $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\vec{a}_3 = \vec{a}_1 + 2\vec{a}_2$ ✓ so the set is lin. dep.!

OR $\vec{0} = \vec{a}_1 + 2\vec{a}_2 - \vec{a}_3$



Definition 2: A set of vectors $\{\vec{a}_1, \vec{a}_2 \dots \vec{a}_M\}$ all in $\mathbb{R}^N$ is linearly dependent if there are $\alpha_1, \alpha_2 \dots \alpha_M \in \mathbb{R}$ (scalar coeffs?) + exclude trivial case not all $\alpha_i = 0$

such that $\sum^M \alpha_m \vec{a}_m = \vec{0}$ ← Lin. combos of all vectors $= \vec{0}$

Are these two defs equivalent? yes!

② what is definition of $\boxed{\text{Linear Independence}}$? NOT lin. dep.

③ True or False: can $\vec{0}$ be in linearly independent set of vectors?

No! eg. $\{\vec{0}, \vec{a}_1, \vec{a}_2 \dots \vec{a}_M\}$

↳ can always write $\vec{0} = 0\vec{a}_1 + 0\vec{a}_2 + \dots + 0\vec{a}_M$ and from definition 1 it is lin. dep!

④ Give an example of 3 linearly independent vectors:

e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ * should be 3D vectors

⑤ Example: Is $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\right\}$ linearly (in)dependent?

3 vectors in 2D space!

one will surely be a lin. combo. of others..

↳ span of first two is $\mathbb{R}^2$

Properties of $A\vec{x} = \vec{b}$ systems of Equations

① $A\vec{x} = \vec{b}$ has a solution iff, $\vec{b} \in$ span (cols of A)
if and only if

② $A\vec{x}$ is a linear combination of the columns of A
(column view)

③ unique solution $\equiv$ unique $\vec{x}$ that satisfies $A\vec{x} = \vec{b}$
is defined as

*NEW ④ Every $\vec{b} \in$ span (cols of A) is uniquely spanned iff the cols of A are Linearly Independent

Possible cases:

$\vec{b} \notin$ span (cols of A) → no solution
not in

$\vec{b} \in$ span (cols of A) → if (cols A) lin. indep. → 1 sol'n

$\vec{b} \in$ span (cols of A) → if (cols A) lin. dep. → ∞ sol'n

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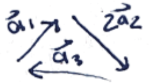

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