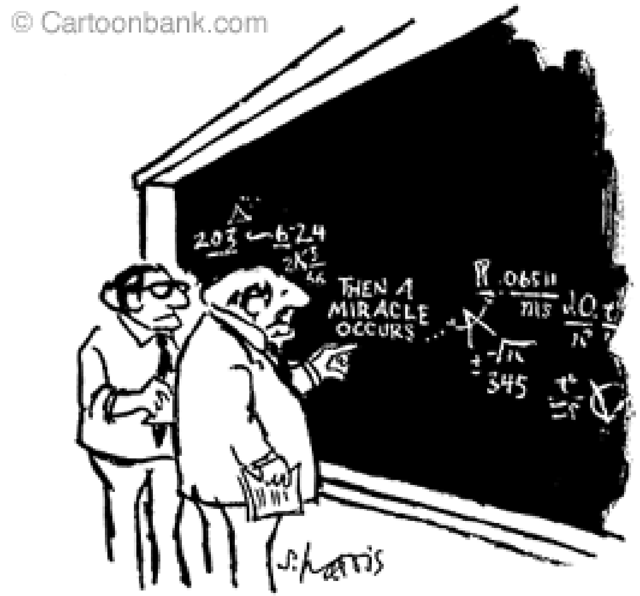


"I think you should be more explicit here in step two."

EE16A Lec2

More Gaussian Elimination, Matrix-Vector Multiplication

Last time:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

A **vector** is an array of numbers

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \boxed{A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

Element 2n of the matrix

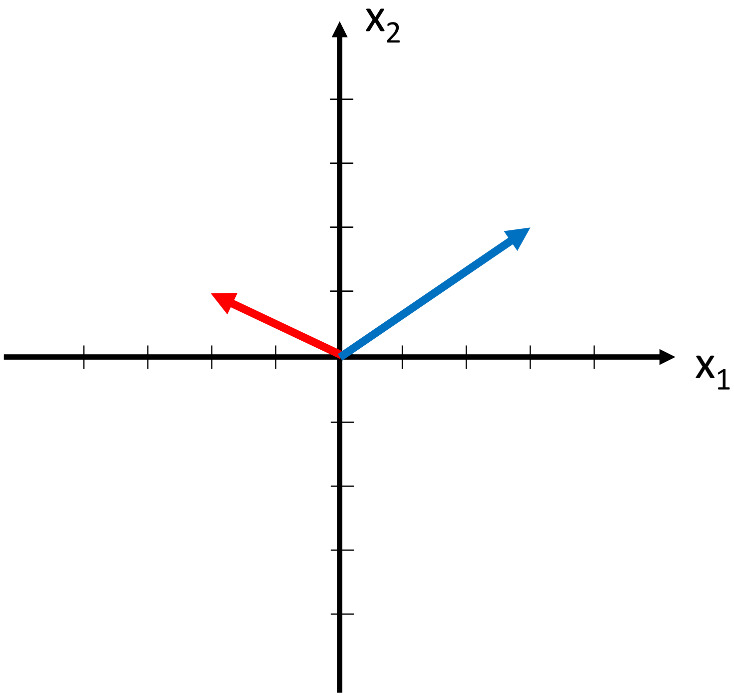
diagonal

A **matrix** is a rectangular arrray of numbers

Drawing vectors graphically

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

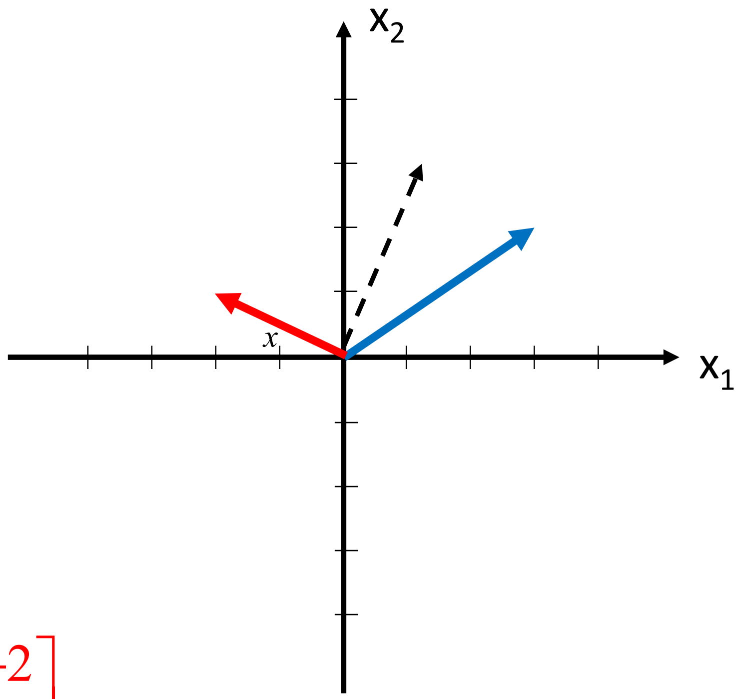
$$\vec{\mathbf{x}}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$



What is the sum of the two vectors?

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$



$$\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

To add vectors, add each corresponding element!

Draw it graphically

Does adding vectors $\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 = \vec{\mathbf{x}}_2 + \vec{\mathbf{x}}_1$?

yes.

Which of these apply?

- ✓ Commutativity: $\vec{x}+\vec{y}=\vec{y}+\vec{x}$
- ✓ Associativity: $\left(\vec{x}+\vec{y}\right)+\vec{z}=\vec{x}+\left(\vec{y}+\vec{z}\right)$
- ✓ Additive identity: $\vec{x}+\vec{0}=\vec{x}$
- ✓ Additive inverse: $\vec{x}+(-\vec{x})=\vec{0}$

Adding matrices

$$\vec{x}_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\vec{x}_1 + \vec{x}_2 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1+0 \\ 3+3 & 4+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 6 & 6 \end{bmatrix}$$

To add matrices, add each corresponding element!

What if they are not same dimensions?

Then you cannot add them.

Vector transpose

$$\vec{\mathrm{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \Rightarrow \vec{\mathrm{x}}^T = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_N \end{bmatrix}$$

T for Transpose!

Matrix transpose

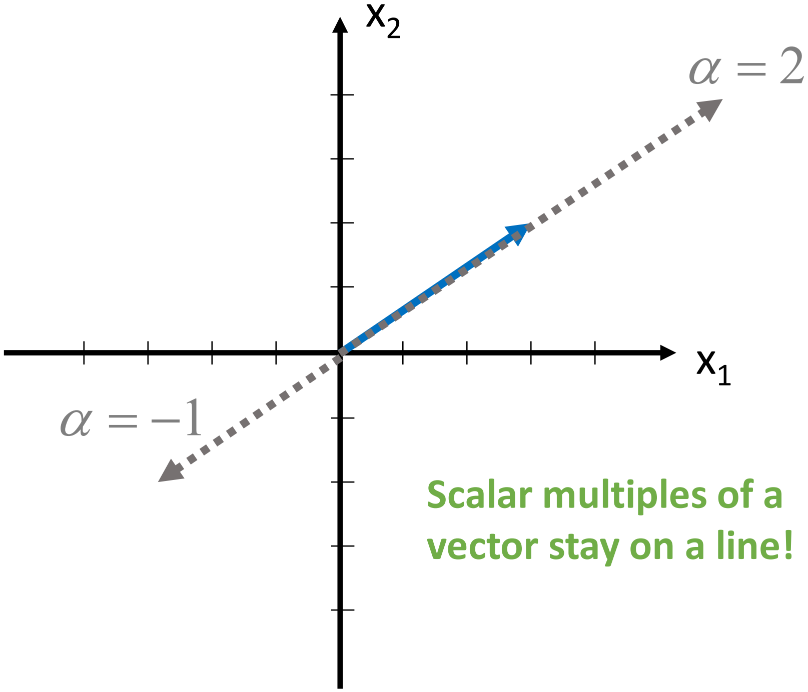
Swaps the rows with the columns

$$\vec{x} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \quad \Longrightarrow \quad \vec{x}^T = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \Longrightarrow \quad \vec{x}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Scaling vectors

$$\vec{\mathrm{x}}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$



Scalar multiples of a vector stay on a line!

What is $\alpha \vec{\mathrm{x}}_1$?

$$\alpha \vec{\mathrm{x}}_1 = \alpha \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha 3 \\ \alpha 2 \end{bmatrix}$$

A vector multiplied by a scalar multiplies all elements of the vector by the scalar.

Scaling matrices

$$\mathbf{x}_1 = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad \in \mathbb{R}^2$$

What is $\alpha \mathbf{x}_1$?

$$\alpha \mathbf{x}_1 = \alpha \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3\alpha & 2\alpha \\ \alpha & 4\alpha \end{bmatrix}$$

A matrix multiplied by a scalar multiplies all elements of the matrix by the scalar.

Multiplying matrices/vectors

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}}_{n \times p} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}}_{m \times p}$$

Must be same!

Multiplying a vector by a vector

$\underset{m \times n}{A}\;\underset{n \times p}{B} = \underset{m \times p}{C}$ $\Rightarrow$ $\therefore$ a row vector ~~can only be~~ multiplied by a column vector

these need to be same!

$\underset{1\times n}{[\;\;]}\;\underset{n \times 1}{[\;\;]} = \underset{1 \times 1}{\square}$

$$\vec{y}^{T}\vec{x} = \underset{1 \times n}{\begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_n \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}} = \underbrace{y_1x_1 + y_2x_2 + y_3x_3 + \cdots y_nx_n}_{\substack{\text{scalar} \\ 1 \times 1}}$$

Also known as "inner product" or "dot product"

$\boxed{\text{Is that same as } \vec{x}\vec{y}^{T}\text{?}}$ No. NOT commutative

Let's try it!

$$\vec{x}\vec{y}^{T} = \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \underset{1 \times m}{\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}} = \underset{n \times m}{\begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_m \\ x_2y_1 & x_2y_2 & \cdots & x_2y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_m \end{bmatrix}}$$

Multiplying a matrix by a vector

$m \times n \quad n \times 1 \quad m \times 1$

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \\ \vdots \quad \ddots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n \end{bmatrix}$$

← scalar

each entry of b is a row-column vector multiplication!

Mathematically:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j$$

$\boxed{\text{Does this explain why } n \text{ must be same for } Ax=b?}$ Yes!

ⓐ This could be re-written as a sum of vectors:
(weighted sum of the columns of A:

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \dots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

New "column" Interpretation

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Example:

$$\begin{bmatrix} -1 & 3 \\ 3 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

unit vector "selects" one column of matrix.

Systems of equations $A\vec{x} = \vec{b}$

$$
\begin{array}{cccc}
\text{‘system’ matrix} & \text{solve for me!} & & \text{measurements} \\
\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} &
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} & = &
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\
m \times n & n \times 1 & & m \times 1
\end{array}
$$

equations/ measurements

unknowns

Row view

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

$$a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n = b_m$$

What do rows represent? How much the variables affect a particular measurement.

Column view

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A x b

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n}_{} = \vec{b}$$

Linear Combination of $\vec{a}$ vectors weighted by the unknowns!

What do columns represent?	How much a particular variable affects all measurements (sensitivity to that variable).
What if one a-vector is zeros?	Then that variable not measured (could be anything)! No unique solution

Multiplying a matrix by a matrix

* take inner product (▭·▯) of each row in A with each column in B (starting from top row of A and leftmost column of B)

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21}) & (a_{11}b_{12}+a_{12}b_{22}) \\ (a_{21}b_{11}+a_{22}b_{21}) & (a_{21}b_{12}+a_{22}b_{22}) \end{bmatrix}$$

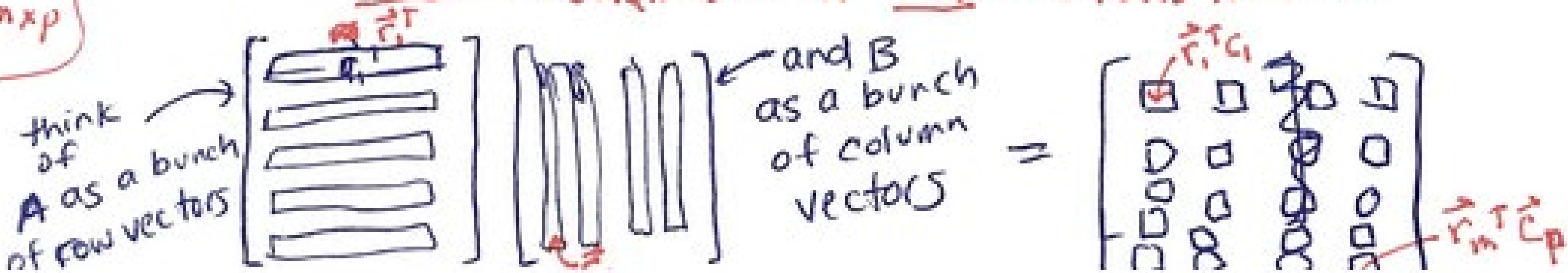
$\boxed{\text{Try it!}}$

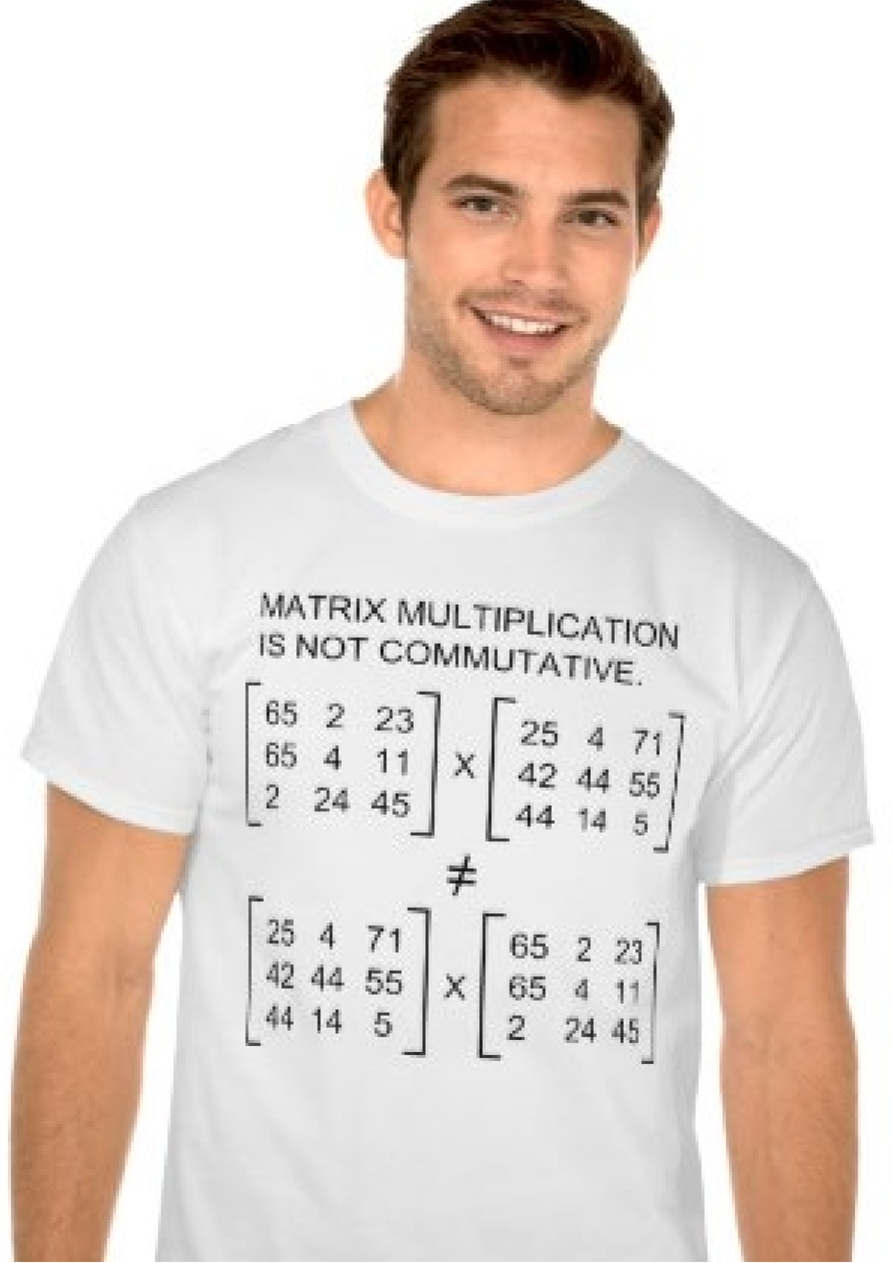
$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0+3(2) & 1(1)+3(3) \\ 2(0)+4(2) & 2(1)+4(3) \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 8 & 14 \end{bmatrix}$$

Remember

$$\underset{m \times n}{A}\ \underset{n \times p}{B} = \underset{m \times p}{C}$$

NOTICE: # columns in A must = # rows in B ✓




MATRIX MULTIPLICATION
IS NOT COMMUTATIVE.
65 2 23
65 4 11
2 24 45
X
25 4 71
42 44 55
44 14 5
≠
25 4 71
42 44 55
44 14 5
X
65 2 23
65 4 11
2 24 45

Recall: Linear systems of equations

$$
\begin{aligned}
a x_1 + a x_2 &= b_1 \\
a x_3 + a x_4 &= b_2 \\
a x_1 + a x_3 &= b_3 \\
a x_2 + a x_4 &= b_4
\end{aligned}
$$

$\Longleftrightarrow$

Can also be represented as:

$$
\left[\begin{array}{cccc|c}
a & a & 0 & 0 & b_1 \\
0 & 0 & a & a & b_2 \\
a & 0 & a & 0 & b_3 \\
0 & a & 0 & a & b_4
\end{array}\right]
$$

=

Or:

$$
\begin{bmatrix} a & a & 0 & 0 \\ 0 & 0 & a & a \\ a & 0 & a & 0 \\ 0 & a & 0 & a \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
$$

Or:

$$Ax = b$$

Recall: *Gaussian Elimination* for solving a linear system of equations

- Goal is to transform your system of equations into ***upper triangular***

$$
\mathbf{A} = \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} & a_{14} \\ 0 & \boxed{a_{22}} & a_{23} & a_{24} \\ 0 & 0 & \boxed{a_{33}} & a_{34} \\ 0 & 0 & 0 & \boxed{a_{44}} \end{bmatrix}
$$

diagonal elements are called *pivots*

- What is allowed:
  - Linear combinations of equations
  - Multiply a row by a scalar
  - Swap rows

- Possible situations:
  - Unique solution
  - Infinitely many solutions (underdetermined)
  - No solution (inconsistent)

Example:

$$\overset{3\times 4}{\begin{bmatrix} 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \overset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \overset{3\times 1}{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}$$

Matrix-Vector form
$A\vec{x} = \vec{b}$

system of equations

$$\boxed{\begin{aligned} 2x_2 + x_3 &= 2 \\ x_1 + x_2 \quad + x_4 &= 3 \\ x_3 &= 1 \end{aligned}}$$

4 unknowns

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

equivalent

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Augmented Matrix form

Want to know whether a system is solvable? (i.e. unique soln)
Do we need to take meas. first, or can we know in design stage?
How to know if our choice of measurements is good?

Let's do Gauss. Elim. on augmented matrix form:
- ↳ scale & add rows (equations)
- ↳ swap rows
- ↳ try to make A "upper triangular"

What if $A_{m\times n}$ is not square? $(m>n) \rightarrow$ more eqn. than unknowns → doesn't mean it's solvable; some may be useless

$(n>m) \rightarrow$ not enough meas.

Looking for A: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$ ← unique sol'n

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 \end{array}\right]$ → infinite sol's

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & \neq 0 \end{array}\right]$ → no sol'n give up!

Note: we didn't use $\vec{b}$ to decide Gauss. Elim. operations
↳ $\boxed{\text{We don't need measurements to know if unique sol'n!}}$

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R1,R2]{swap} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

eliminate x_1 done!

Next, normalize this

$$\xrightarrow{R_2/2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Jargon Alert!

This is called $\boxed{\text{Row Echelon form}}$ → as close as possible to upper triangular if A not square

↳ all non-zero rows are above zero rows

↳ leading coefficient is to right of that in row above

first non-zero entry in row from left → right

↳ sometimes: leading coeffs. are 1

Now we want to put in $\boxed{\text{Reduced Row Echelon form}}$ (RREF)

mmm, jargon!

* Basically means you've done back-substitution

↳ each column w/ leading 1 has zeros everywhere else

↳ rows can have more #s!

leading coeffs.

next, deal w/ this guy!

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R2-R3]{R1-R2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

rest of col zero ✓

rest of col NOT zero! not RREF

$\xrightarrow{R1+R3}$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

* Just because RREF doesn't mean solvable!

↑ ↑ ↑

all zeros except lead. coef.! ☺ RREF!

$\boxed{\text{Pivot}}$ - the leading coeffs once in RREF

What is up with this guy?

↳ "Free variable" has no leading coeff. → can set it to find one of inf. sol's

So x_1, x_2, x_3 are $\boxed{\text{"basic variables"}}$

x_4 is $\boxed{\text{"free variable"}}$

In this example, $x_3 = 1$, $x_2 = 0$, $\underbrace{x_1 + x_4 = 3}$

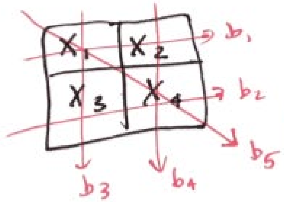
so we can pick $x_4 = t$

so that $x_1 = 3 - t$

$\rightarrow \vec{x} = \begin{bmatrix} 3-t \\ 0 \\ 1 \\ t \end{bmatrix}$. this is called a $\boxed{\text{parametric solution}}$

once you pick t, rest is solved, but any t solves system.

Let's go back to tomography example:



$$\boxed{\begin{aligned}
x_1 + x_2 \quad\quad\quad\quad &= b_1 \quad ① \\
x_3 + x_4 &= b_2 \quad ② \\
x_1 \quad + x_3 \quad\quad &= b_3 \quad ③ \\
x_2 \quad\quad + x_4 &= b_4 \quad ④
\end{aligned}}$$

this was not enough

so add

$$x_1 \quad\quad\quad + x_4 = b_5 \quad ⑤$$

In augmented matrix form:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 1 & b_2 \\ 1 & 0 & 1 & 0 & b_3 \\ 0 & 1 & 0 & 1 & b_4 \end{array}\right]$$

How to know if this is good set of measurements?

(you do G.E. on own 1min)

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & 1 & b_4 \\ 0 & 0 & 1 & 1 & b_4 + b_3 - b_1 \\ 0 & 0 & 0 & 0 & b_2 - b_4 - b_3 + b_1 \end{array}\right]$$

if $= 0 \to$ infinite sol's

if $\neq 0 \to$ insonsistent

but we don't need meas. to know ~~whe~~ there is <u>no unique sol'n</u>!

↳ depends only on system.

What if i use $b_1, b_2, b_3, \underline{b_5}$?

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 1 & b_2 \\ 1 & 0 & 1 & 0 & b_3 \\ 1 & 0 & 0 & 1 & b_4 \end{array}\right] \xrightarrow{\text{G.E.}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & b_1 - b_3 \\ 0 & 0 & 1 & -1 & b_3 - b_5 \\ 0 & 0 & 0 & 1 & \frac{b_2 + b_5 - b_3}{2} \end{array}\right]$$

new! eqn! →

didn't depend on actual $\vec{b}$, but on <u>SYSTEM</u>

upper triangular / REF

↳ does give unique sol'n!