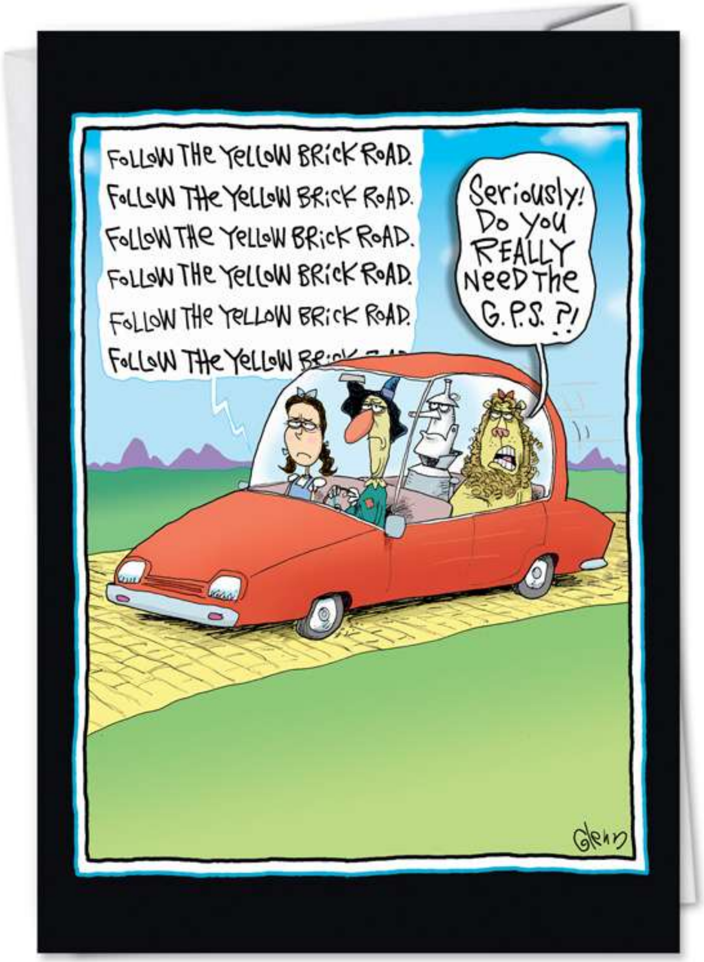


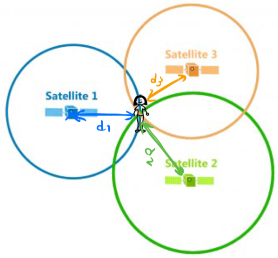
EECS 16A

Correlation and Classification

Admin

- Midterm 2 done! Yay!
- Redo due ~~Monday~~ – must complete to be eligible for clobber
  FRIDAY!!!

Last time: $\boxed{\text{GPS `trilateration'}}$ we can find our position in 2D world



→ if we know our distance to 3 satellites
+ the satellite positions are known
+ the satellites are <u>not</u> colinear

<u>Measuring distance to a satellite:</u>

↳ The satellite is constantly putting out a 'song' (signal) played on repeat (one-way transmit)

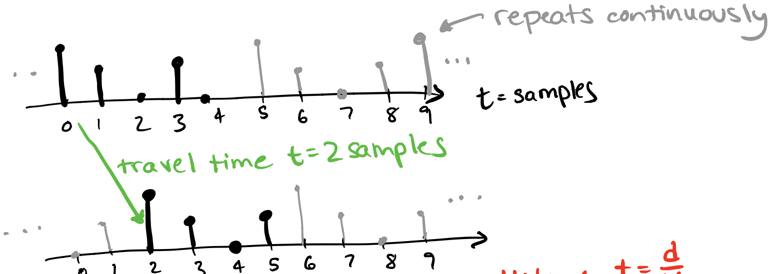
↳ our reciever has a synchronized clock and tries to figure out how delayed the signal is due to travel time from the satellite → reciever

distance from satellite → $d = v\,t$ ← time delay = travel time

for GPS $v = 3\times10^8$ m/s

<u>Example:</u>

signal from satellite:



recieved signal:

shift/delay is proportional to distance $t = \frac{d}{v}$

signal (song, gold code) $\vec{s} = \begin{bmatrix}2\\1\\0\\1\end{bmatrix}$ recieved signal: $\vec{r} = \begin{bmatrix}0\\1\\2\\1\end{bmatrix}$

all possible shifted vectors: $\vec{s}_0 = \begin{bmatrix}2\\1\\0\\1\end{bmatrix}$ $\vec{s}_1 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}$ $\vec{s}_2 = \begin{bmatrix}0\\1\\2\\1\end{bmatrix}$ $\vec{s}_3 = \begin{bmatrix}1\\0\\1\\2\end{bmatrix}$

shifted by 0 samples; shifted by 1 sample

Since $\vec{r} = \vec{s}_1$, the delay (shift) was 2 samples

Next, we wanted a robust way to find which of the shifted vectors is most similar to $\vec{r}$.

↳ What is k when $\|\vec{r} - \vec{s}_k\|^2$ is minimum?

$$\boxed{\text{Norm of a vector } \vec{x} \text{ is } \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots x_N^2}}$$

↳ length, magnitude

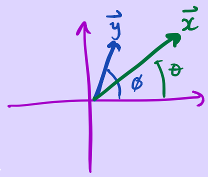
Then, we calculated that $\|\vec{r} - \vec{s}_k\|^2$ is minimum when $\underbrace{\vec{r}^T\vec{s}_k}_{\text{inner product!}}$ is maximum

Inner Product of vectors $\underset{n\times 1}{\vec{x}}$ and $\underset{n\times 1}{\vec{y}}$ is $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T\vec{y}$

↳ dot product, correlation

$$= x_1 y_1 + x_2 y_2 + \dots x_N y_N \quad {}_{1\times 1}$$

$$= \sum_{i=1}^{n} x_i y_i$$



$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \, \|y\| \cos\underbrace{(\theta - \phi)}_{\text{measures how aligned/similar two vectors are}}$$

Cauchy-Schwartz Inequality $\boxed{\langle \vec{x}, \vec{y} \rangle \leq \|\vec{x}\| \, \|\vec{y}\|}$

Properties of inner products:

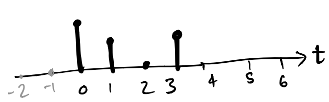
1. $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$
2. $\langle \vec{a} + \vec{x}, \vec{y} \rangle = \langle \vec{a}, \vec{y} \rangle + \langle \vec{x}, \vec{y} \rangle$
3. $\langle \alpha \vec{x}, y \rangle = \alpha \langle \vec{x}, y \rangle$
4. $\langle \vec{x}, \vec{x} \rangle \geq 0$ (unless $\vec{x} = 0$ then $\langle \vec{x}, \vec{x} \rangle = 0$)

So, to find the shift, we need to compute inner products of the recieved signal with all possible shifts of the sent signal from the satellite. The values of these innerproducts arranged into a new signal is the cross-correlation:

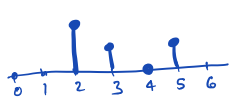
Cross-correlation defined as $\underbrace{\text{corr}_{\vec{r}}(\vec{s})}_{\text{correlation of } \vec{r} \text{ with } \vec{s}} \overset{\text{corr value at element } k}{[k]} = \sum_{i=0}^{\infty} r(i)\, s\,(i-k)$

k is the shift index!

$\vec{s} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$



$\vec{r} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$



↳ basically, each value in the correlation function is the inner product for a given shift value!

↳ largest value tells you what shift is!

Example:

skip undefined/zero terms since assumed zero

$$\text{corr}_{\vec{r}}(\vec{s})[0] = r(2)s(2)+r(3)s(3)+r(4)\cancel{s(4)}+r(\cancel{5})s(5) = 2(0)+1(1) = 1$$

$$\text{corr}_{\vec{r}}(\vec{s})[1] = r(2)s(1)+r(3)s(2)+r(4)s(3)+r(\cancel{5})s(4) = 2(1)+(1)0+0(1) = 2$$

$$\text{corr}_{\vec{r}}(\vec{s})[2] = r(2)s(0)+r(3)s(1)+r(4)s(2)+r(5)s(3) = 2(2)+1(1)+0(0)+1(1) = 6$$

$$\text{corr}_{\vec{r}}(\vec{s})[3] = r(2)\cancel{s(-1)}+r(3)s(0)+r(4)s(1)+r(5)s(2) = 1(2)+0(1)+1(0) = 2$$

$$\text{corr}_{\vec{r}}(\vec{s}) = \begin{bmatrix} 1 & 2 & 6 & 2 \end{bmatrix}$$

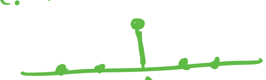
largest value is in [2] position, so the shift is 2

Why is the inner product a good metric for which shift is correct?

- noise robust (works even when dog barks)
- can handle attenuation (if song is quieter than expected)
- will work well even when multiple satellites 'on'! (reciever hears linear combo. of multiple songs)

What signals make good songs?

- want shifted versions to be uncorrelated i.e. ideal $\text{corr}_{\vec{r}}(\vec{s})$ is
  - orthogonal? (almost)



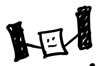
Ex. $\vec{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ × BAD

$\vec{s} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ × BAD (repeats)

$\vec{s} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$ ✓ good

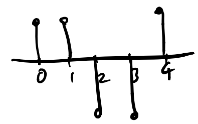
Satellite classification: Which satellite am i talking to?

Example:



Satellite A

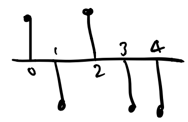
Signature: $\vec{s}_A = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ ← uses only 1 and −1 (like morse code)



"Gold code"

$\|\vec{s}_A\| = \sqrt{5}$

$\vec{s}_B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$



$\|\vec{s}_B\| = \sqrt{5}$

2 vectors in 5D space at some angle.

Say i receive $\vec{r}$ and want to know which satellite it's from:

$$\vec{r} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$ ← this is $\vec{s}_A$ shifted by 0 samples

But in real life, there is noise, so we might recieve something more like this

$$\vec{r} = \begin{bmatrix} 0.9 \\ 1.1 \\ -1.2 \\ 0 \\ 1 \end{bmatrix} = \underset{\text{signal}}{\vec{s}} + \underset{\text{noise}}{\vec{n}}$$ ← need to classify in the presence of noise

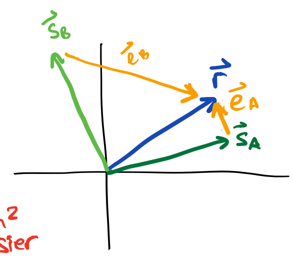
↳ may be $\vec{s}_A$, $\vec{s}_B$ or a linear combo

If just one satellite: Want to know if $\vec{r}$ is "closer" to $\vec{s}_A$ or $\vec{s}_B$?

↳ <u>Classify</u> with noisy signal (classic problem)

Which signal $\vec{s}_A$ or $\vec{s}_B$ is "closest" to $\vec{r}$?

$\vec{e}_A = \vec{r} - \vec{s}_A$ $\quad \vec{e}_B = \vec{r} - \vec{s}_B$



Find satellite for which $\vec{e}$ is minimized in terms of the norm: Find which $\vec{s}_i$ minimizes $\|\vec{e}_n\|^2$ ← norm² is easier

↳ satellite index (i.e. A or B)

$$\underbrace{\underset{\text{over all satellites}}{\text{minimize}} \|\vec{e}_n\|^2}_{\text{optimization problem!}} = \langle \vec{e}_n, \vec{e}_n \rangle = \vec{e}_n^T \vec{e}_n$$

$$= (\vec{r} - \vec{s}_n)^T (\vec{r} - \vec{s}_n)$$

$$= (\vec{r}^T - s_n^T)(\vec{r} - \vec{s}_n)$$

$$= \vec{r}^T \vec{r} + \vec{s}_n^T \vec{s}_n - \vec{s}_n^T \vec{r} - \vec{r}^T \vec{s}_n$$

$$= \|\vec{r}\|^2 + \|\vec{s}_n\|^2 - \langle \vec{s}_n, \vec{r} \rangle - \langle \vec{r}, \vec{s}_n \rangle$$ same thing

$$= \underset{\uparrow \text{fixed}}{\|\vec{r}\|^2} + \underset{\uparrow \text{fixed (all have equal norm)}}{\|\vec{s}_n\|^2} \boxed{- 2 \langle \vec{s}_n, \vec{r} \rangle}$$

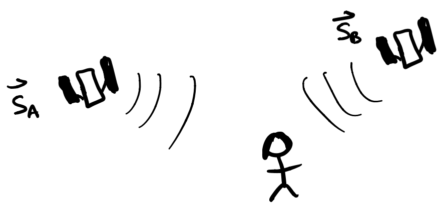
↳ only term that changes with n

So the problem amounts to minimizing $-2\langle \vec{s}_n, \vec{r}\rangle$ over all n

equivalent ↳ maximize $2\underbrace{\langle \vec{s}_n, \vec{r}\rangle}_{\text{collinear} \to \max \langle,\rangle}$ over all n

Algorithm: For all satellites $\vec{s}_n$, compute $\langle \vec{r}, \vec{s}_n\rangle$. Find largest value.

Multiple satellites transmitting at once:



Each is sending its 'song' via EM waves ↳ obey principle of superposition!

$$\vec{r} = \vec{s}_A + \vec{s}_B + \vec{n}$$

↖ noise

We can check inner prod. to see how similar recieved signal is to $\vec{s}_A$:

$$\langle \vec{r}, \vec{s}_A\rangle = \langle \vec{s}_A + \vec{s}_B + \vec{n}, \vec{s}_A\rangle$$

satellite A is 'on'

$$= (\vec{s}_A + \vec{s}_B + \vec{n})^T \cdot \vec{s}_A$$

$$= \vec{s}_A^T \cdot \vec{s}_B + \vec{s}_B^T \cdot \vec{s}_A + n^T \cdot \vec{s}_A$$

$$= \underbrace{\langle \vec{s}_A, \vec{s}_A\rangle}_{\text{large}} + \underbrace{\langle \vec{s}_B, \vec{s}_A\rangle}_{\text{small by design}} + \underbrace{\langle \vec{n}, \vec{s}_A\rangle}_{\text{small}}$$

↖ inner products of $\vec{s}_A$ w/ $\vec{s}_A, \vec{s}_B, \vec{n}$

Now try $\langle \vec{r}, \vec{s}_c\rangle = \langle \vec{s}_A + \vec{s}_B + \vec{n}, \vec{s}_c\rangle$

Satellite C is not 'on'

$$= \underbrace{\langle \vec{s}_A, \vec{s}_c\rangle}_{\text{small}} + \underbrace{\langle \vec{s}_B, \vec{s}_c\rangle}_{\text{small by design}} + \underbrace{\langle \vec{n}, \vec{s}_c\rangle}_{\text{small}}$$

↳ nothing is correlated w/ random noise

Can set a threshold for detecting $\vec{s}_A$:

if $\langle \vec{r}, \vec{s}_A\rangle \geq$ threshold, then $\vec{s}_A$ detected!

Does the shift matter?

What signals make good songs?

↳ want shifts to be uncorrelated (at all shifts) AND two songs to be uncorrelated

Ex. $\vec{s}_A = \begin{bmatrix}1\\2\\3\end{bmatrix}$ $\vec{s}_B = \begin{bmatrix}2\\3\\1\end{bmatrix}$ X BAD! (one is shifted version of other)