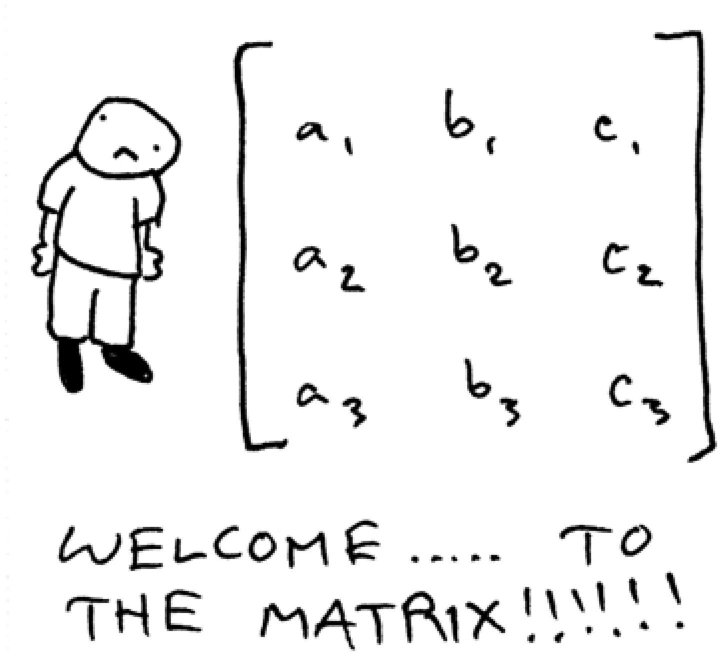
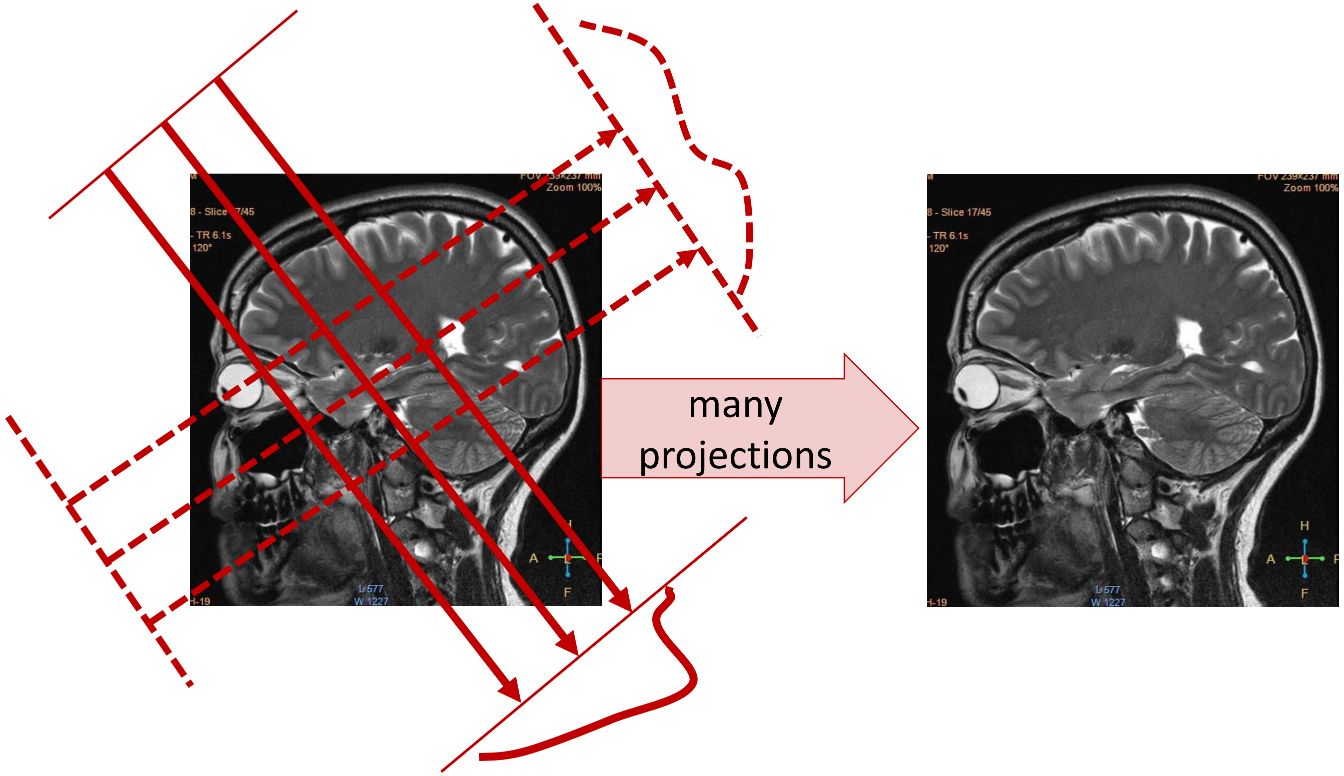


EE16A Lec1

Systems of Linear Equations and Gaussian Elimination

Last time: Tomography



If you like tomography, you'll love EE123, EE145B, EECS261, EE225E!

Or research with profs:
Miki Lustig
Chunlei Liu

What is a projection?

Sum of values along a line.

Vectors are arrays of numbers

represents a SINGLE POINT in N-dimensional space

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

column vector

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_N \end{bmatrix}$$

row vector

What are the dimensions of this vector? $\vec{x} \in \mathbb{R}^3$

3-dimensional vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

A matrix is a rectangular array of numbers

$$\text{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \boxed{A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

This is element (component) 2n of the matrix

What are the dimensions of A?

m rows and **n columns** means it is a **m x n** matrix

Some special types of matrices

zero matrix

$$\vec{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

diagonal matrix

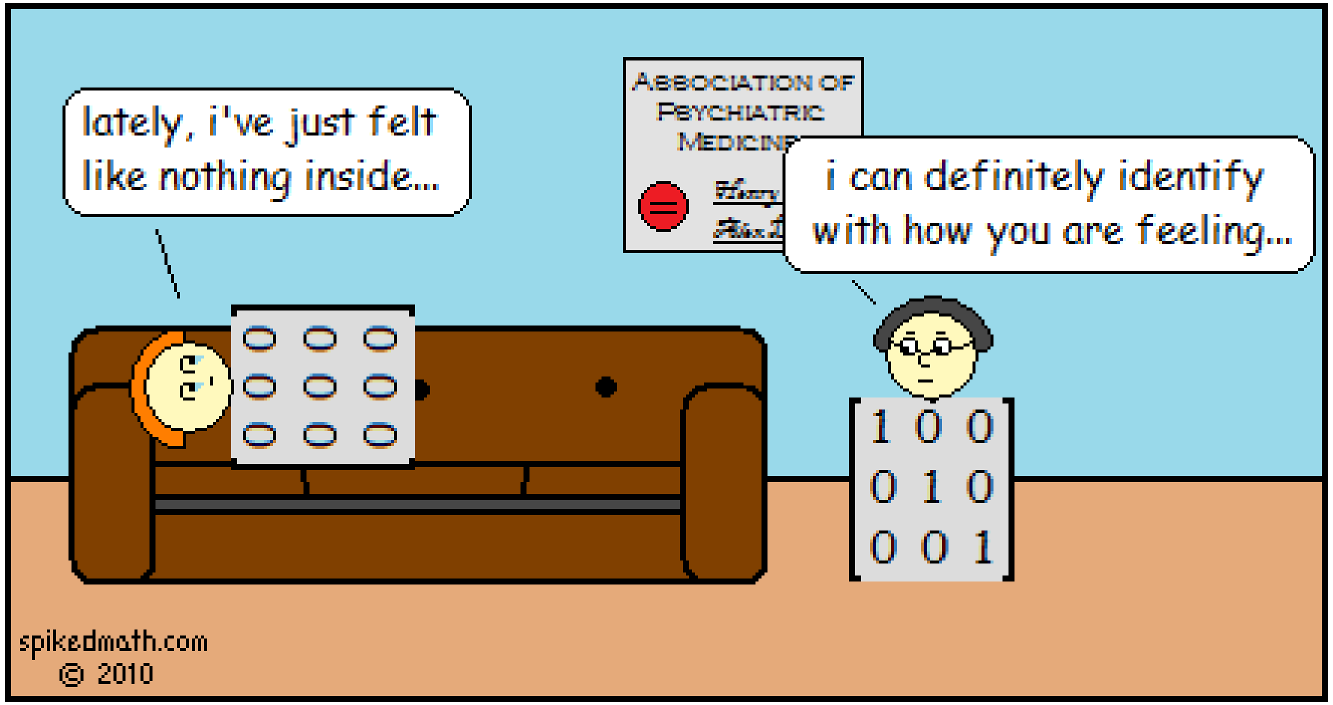
$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

upper triangular matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$


lately, i've just felt like nothing inside...
Association of Psychiatric Medicine
i can definitely identify with how you are feeling...
1 0 0
0 1 0
0 0 1
spikedmath.com
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Ways of representing linear systems of equations

$$
\begin{aligned}
a x_1 + a x_2 \qquad\qquad\qquad &= b_1 \\
a x_3 + a x_4 &= b_2 \\
a x_1 \qquad + a x_3 \qquad\qquad &= b_3 \\
a x_2 \qquad + a x_4 &= b_4
\end{aligned}
$$

Can also be represented as:

$$
\left[\begin{array}{cccc|c}
a & a & 0 & 0 & b_1 \\
0 & 0 & a & a & b_2 \\
a & 0 & a & 0 & b_3 \\
0 & a & 0 & a & b_4
\end{array}\right]
$$

Or:

$$
\begin{bmatrix} a & a & 0 & 0 \\ 0 & 0 & a & a \\ a & 0 & a & 0 \\ 0 & a & 0 & a \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
$$

Or:

$$Ax = b$$

The lazy way

Today: Solving a linear system of equations

First, write in simple form:

$$
\begin{aligned}
&① \quad x + 4y = 6 \\
&② \quad -y + 2x = 3
\end{aligned}
\quad \Longleftrightarrow \quad
\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & -1 & 3 \end{array}\right]
$$

Don't forget to put x in one column and y in another

Now solve it. How?

Start plugging equations into each other….
See what happens?

e.g.
1) Solve ① for x and plug into ②

2) 4 x ② + ①

$$
\left.\begin{aligned}
&① \quad x + 4y = 6 \\
&② \quad 2x - y = 3
\end{aligned}\right\} \text{ Solving example } \quad 4 \times ② + ① \rightarrow
\begin{array}{r}
\left(8x - \cancel{4y} = 12\right. \\
+ \quad \left.x + \cancel{4y} = 6\right) \\ \hline
9x \quad = 18
\end{array}
$$

$$\boxed{x = 2}$$

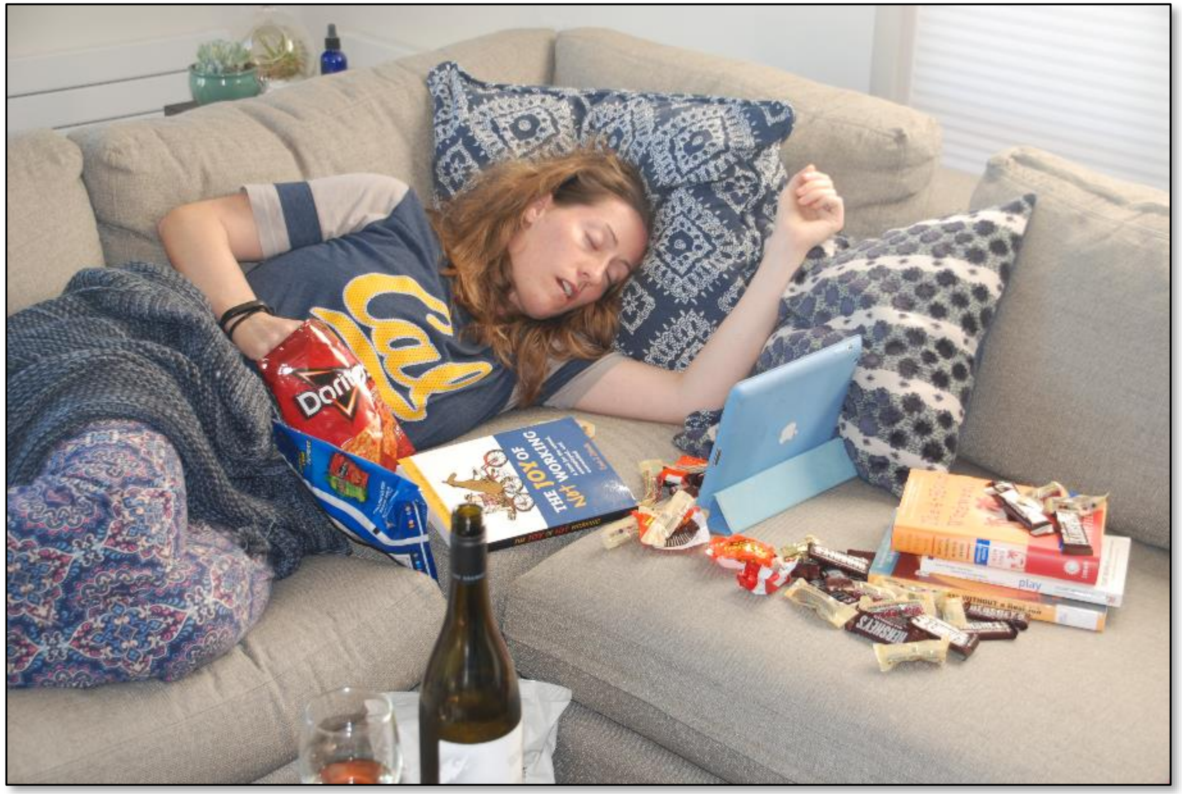
plug into ①:

$$(2) + 4y = 6$$

$$4y = 4$$

$$\boxed{y = 1}$$

GOAL: to develop a systematic way of solving systems of equations with clear rules that *can be done by a computer*



(then I can be even lazier)

Gaussian Elimination for solving a linear system of equations

- Specifies the order in which you combine equations (rows) to “eliminate” (make zero) certain elements of the matrix

- Goal is to transform your system of equations into ***upper triangular***

$$\mathbf{A} = \begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} & a_{14} \\ 0 & \boxed{a_{22}} & a_{23} & a_{24} \\ 0 & 0 & \boxed{a_{33}} & a_{34} \\ 0 & 0 & 0 & \boxed{a_{44}} \end{bmatrix}$$

diagonal elements are called ***pivots***

① $x + 4y = 6$

② $2x - y = 3$

in simple form:

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & -1 & 3 \end{array}\right]$$

$$\begin{array}{c} \\ ② - 2① \end{array} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & -9 & -9 \end{array}\right]$$

we 'eliminated' x to make this UPPER TRIANGULAR

↳ can now read off bottom row variable

$$-9y = -9$$

$$\boxed{y = 1}$$

then plug into top row

$$x + 4y = 6$$

$$x + 4 = 6$$

$$\boxed{x = 2}$$

The Gauss. elim. way to solve is just a specific

② − 2①

$$\underbrace{2x - y}_{②} - 2\underbrace{(x + 4y)}_{①} = \underbrace{3}_{②} - 2\underbrace{(6)}_{①}$$

$$-9y = -9$$

$$\boxed{y = 1}$$

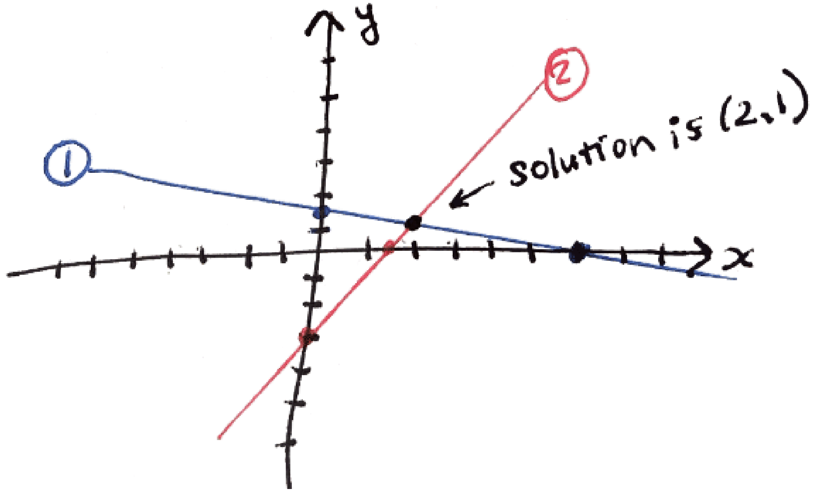
use ① to "eliminate" x from ②

then plug it in, say to ①:

$$x + 4(1) = 6$$

$$\boxed{x = 2}$$

could also solve graphically: (find x, y intercept)



3D example | Could you solve this graphically? Intersection of 3 planes. Good luck with that!

$$2y + z = 1 \quad (1)$$
$$2x + 6y + 4z = 10 \quad (2)$$
$$x - 3y + 3z = 14 \quad (3)$$
$$\rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 2 & 6 & 4 & 10 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Now follow the same procedure:
→ use 1st eqn to eliminate 1st variable from 2nd eqn.

Hmm... But it's a ZERO! What to do?

we are allowed to swap rows, so swap Row 1 & 2

giving
$$\left[\begin{array}{ccc|c} 2 & 6 & 4 & 10 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Now what? divide 1st row by 2
Why is that allowed? doesn't change equation.
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Now what? subtract Row 1 from Row 3
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{array}\right]$$

Now how many letters have I saved myself from writing?!

zeros! Good

What to make zero (eliminate) first? the -6

Which row should I use? Row 2 because if we use Row 1 we'll lose a zero (= BAD!)

PIVOTS

upper triangular ☺
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 4 & 12 \end{array}\right] \rightarrow 4z = 12 \rightarrow \boxed{z = 3}$$

To be systematic, we should instead make matrix manipulation so that 3rd row pivot is 1:

divide Row 3 by 4
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

ELIMINATION PART

Now we read off $z = 3$

For the "Plug in" part, now we need to back-substitute upwards from bottom:

e.g. plug $z = 3$ into row 2 → $2y + 1(z) = 1$
$$2y + 3 = 1$$
$$\boxed{y = -1}$$

then plug $z = 3$ and $y = -1$ into Row 1
$$x + 3y + 2z = 5$$
$$x + 3(-1) + 2(3) = 5$$
$$x = 5 + 3 - 6 = \boxed{2 = x}$$

To translate this into matrix manipulations:

(Row 2 - Row 3)
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

divide Row 2 by 2
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
Now we read off $y = -1$ ✓

Finally, (Row 1 - 3Row 2 - 2Row 3)
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
Read off $x = 2$ ✓

What is allowed in Gaussian elimination?

- Linear combinations of equations (adding scalar multiples of rows to other rows)
- Multiply a row by a scalar
- Swap rows

Goals of Gaussian Elimination algorithm

- Equation with i^{th} variable in the i^{th} row
- Coefficient of the i^{th} variable in the i^{th} row becomes 1
- For rows j=i+1 and higher, subtract row i times the entry in (j,i) to cancel variable i

Gaussian elimination was part of the work of human computers

What might be the variables/measurements in calculating rocket trajectories?

Position, direction of motion, tilt, power/thrust, weight…

Will it always work?

Example:

$x + 4y = 6$ ①
$2x + 8y = 12$ ②

$2① = ②$
No new info!

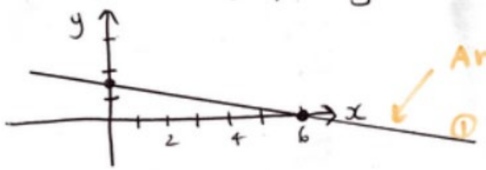
$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right] \xrightarrow[\text{Gauss. Elim.}]{\text{try}} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$

pivot is zero!

$0x + 0y = 0$

Can you solve this? No. (1 equation, 2 unknowns)

Let's look at it graphically:



Any point on this line is a solution

Questions:

Is there a situation where infinitly many solutions is a good thing?
↳ Yes. In design, gives flexibility.

If you can't solve → UNDER DETERMINED, what should you do?
↳ take more meas!?

Example 2:

$x + 4y = 6$
$2x + 8y = 10$

$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 10 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & -2 \end{array}\right]$

Pivot 0

$0 = -2$
cannot Solve!

Inconsistent
NO SOLUTION!

Let's look at it graphically:

parallel lines!
No intersection ≡ no solution

In both cases (~~???~~ under-determined, + inconsistent), the zero pivot was a red flag!

Try

$x + 4y = 6$
$2x + 8y = 12$

$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right]$

1) no need to normalize

2) $R2 - 2R1$ $\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$

$0x + 0y = 0$
Infinite sol'ns

In both cases, the number in the pivot position* being zero was a red flag! (*technically, it's not called a pivot if zero)

Possible situations

- Unique solution
- Infinitely many solutions (underdetermined)
- No solution (inconsistent)

Is it possible to have exactly 2 solutions?

No. consider graphically: two lines cannot intersect in exactly two places

Cats vs. Dogs

These measurements are different linear combinations of two images.

measurements

Can you guess what the measurements are?

Top: 0.6 (dog) + 0.4 (cat)
Bottom: 0.6 (cat) + 0.4 (dog)

Can I solve for both images from just these two linearly combined images? Just one? None? How many images do I need minimum?

Two images is enough if they're linearly independent at each pixel!

300x300 pixels = 90,000

Cats vs. Dogs

measurements

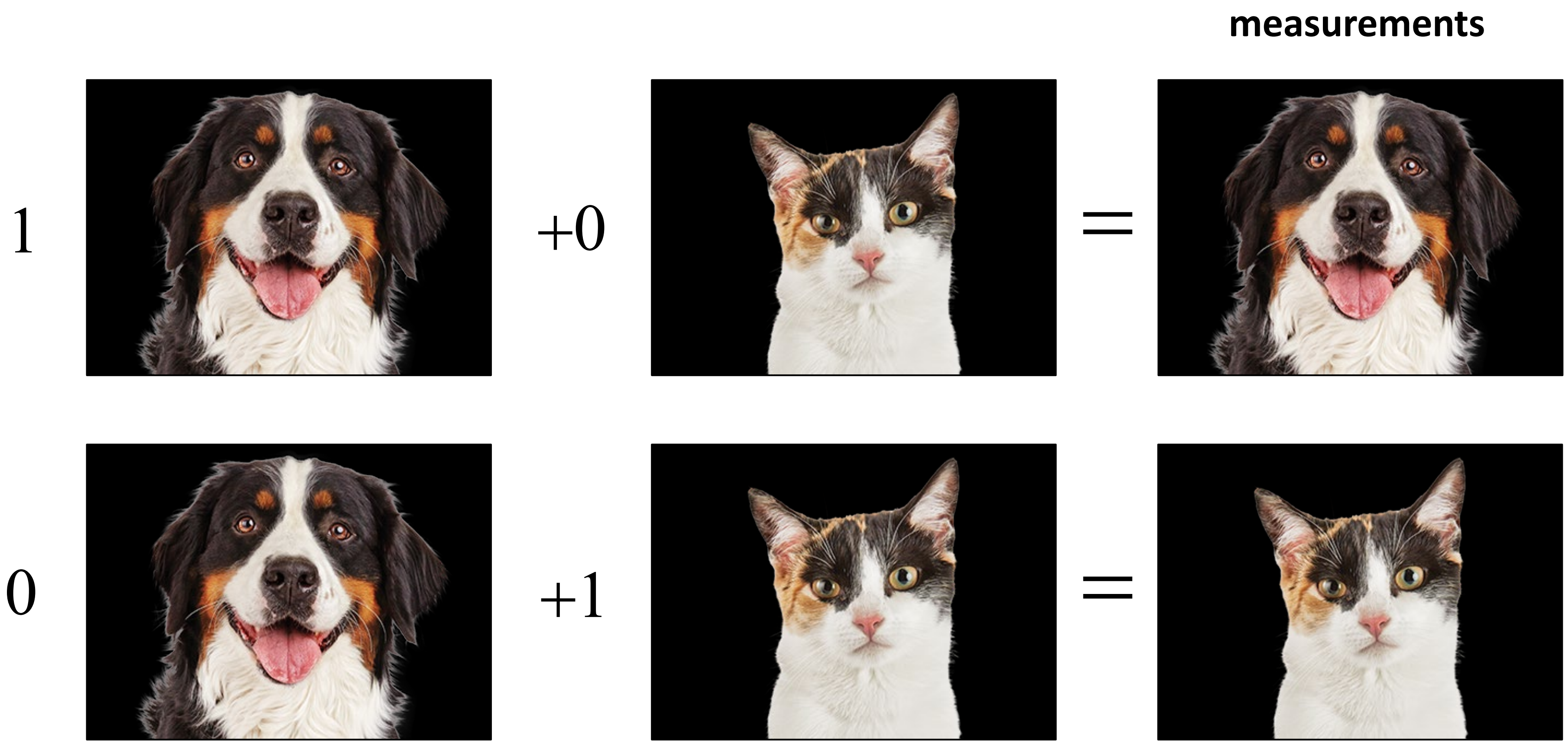
0.6  +0.4  = 

0.4  +0.6  = 

What are the ideal measurements?

Depends. Maybe direct measurements of cat and dog…

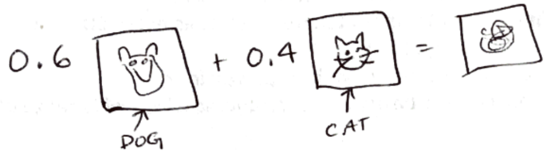
Cats vs. Dogs: Direct measurements



Very easy to solve!

CATS vs. DOGS

How would you set up the Linear system of equations?

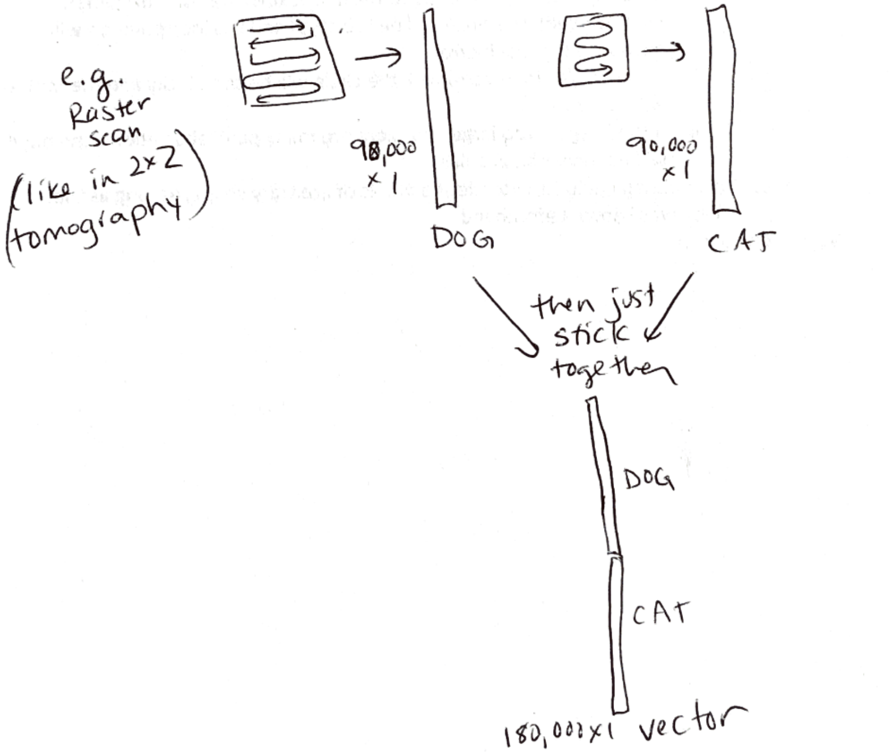


What are unknowns? ALL pixels of DOG image **and** ALL pixels of CAT image!

i.e. (300 pixels x 300 pixels) x2

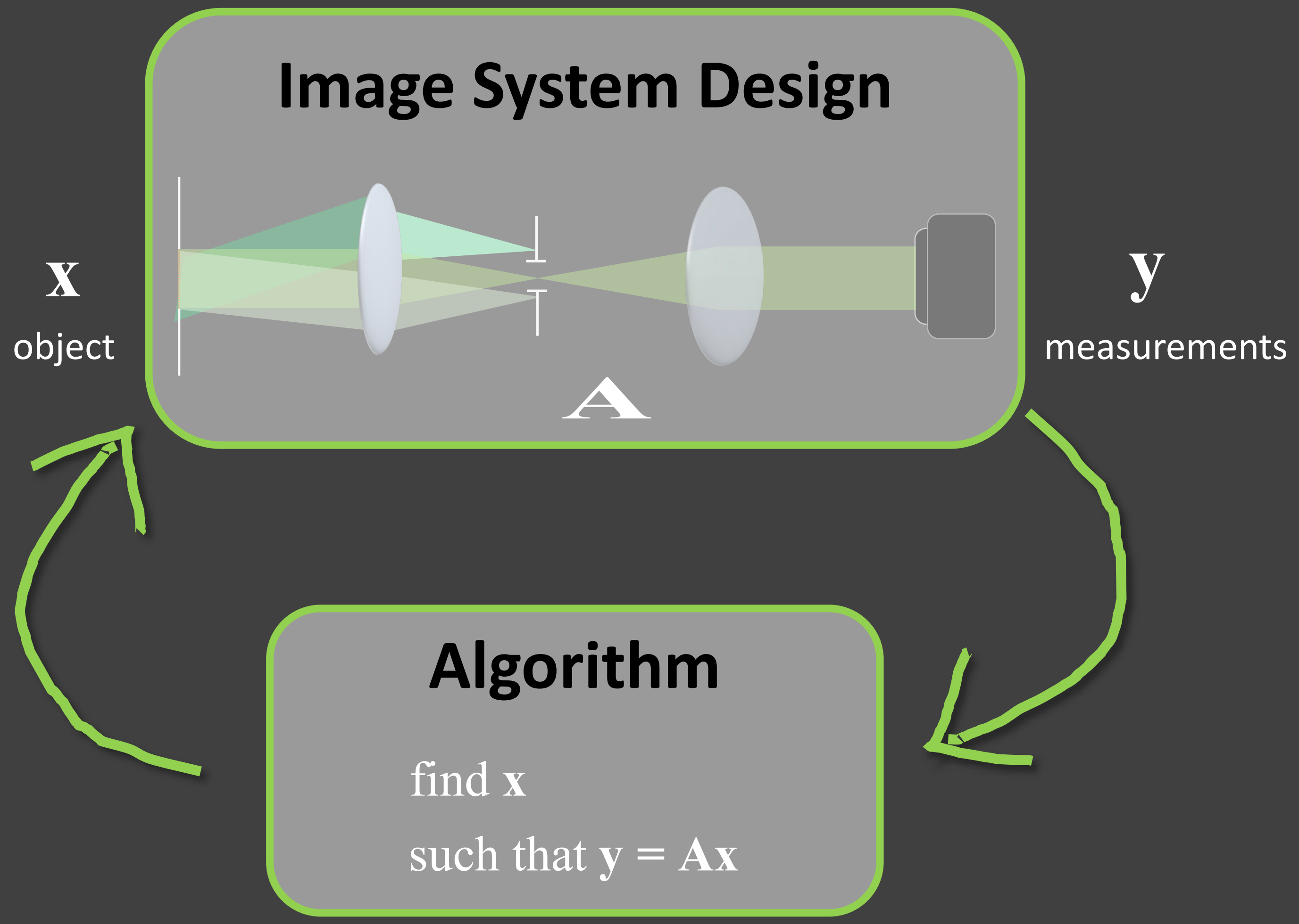
= 180,000 pixels

Need to put them all into one vector:

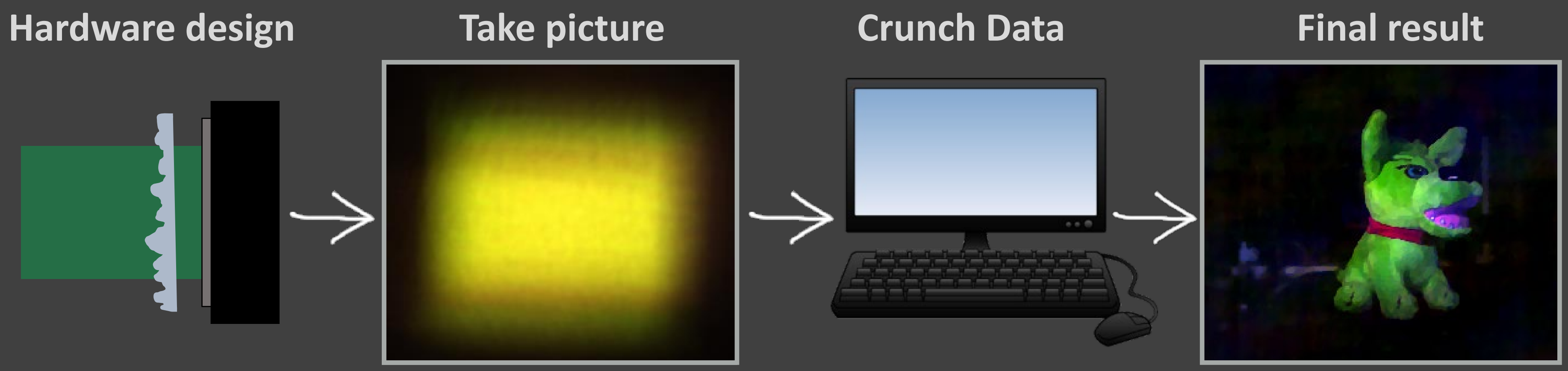


My Research uses linear algebra

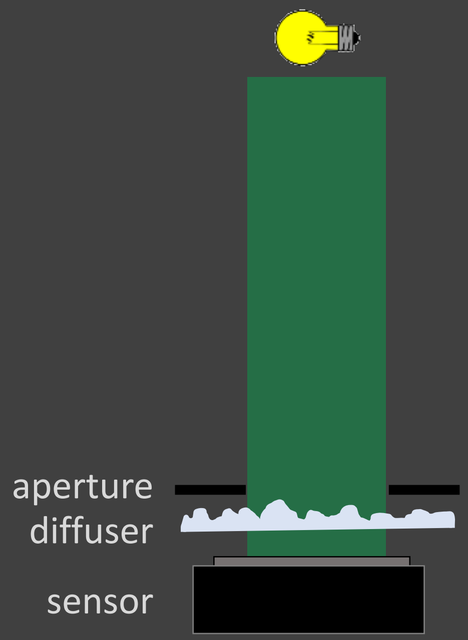
Computational Imaging: joint design of hardware and software



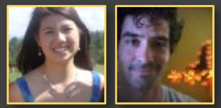
Computational imaging pipeline



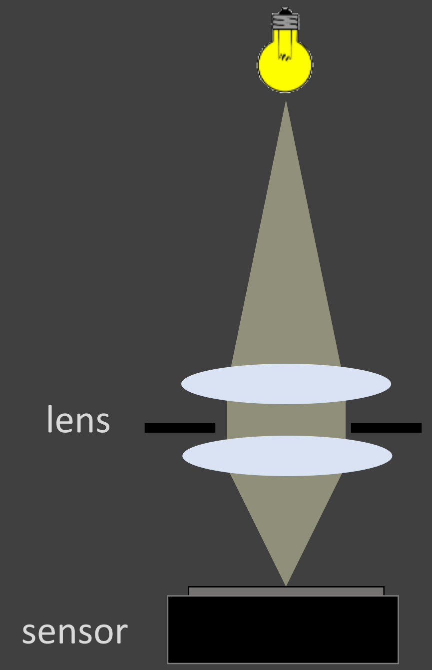
DiffuserCam: tape a diffuser onto a sensor

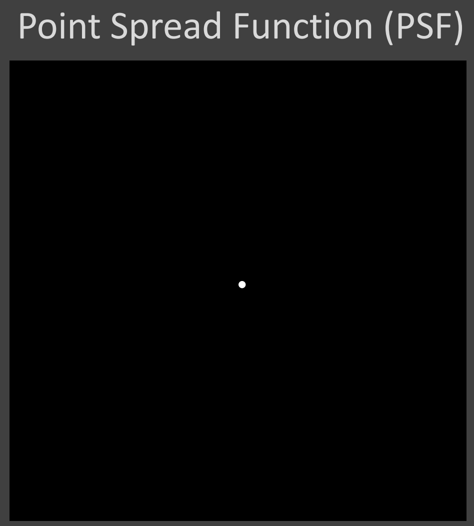


Grace Kuo
Nick Antipa

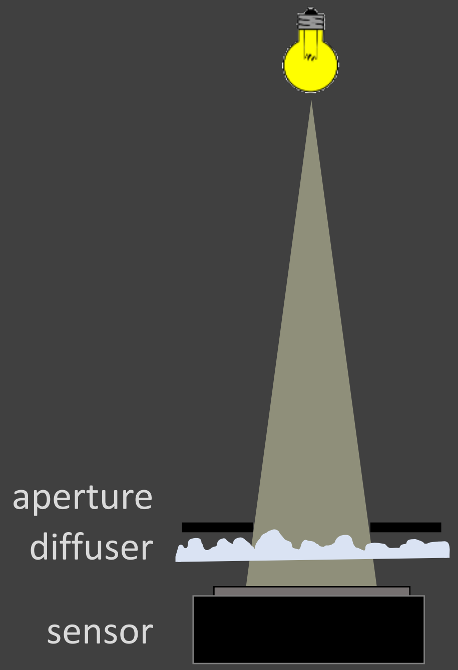


Lenses map a point to a point

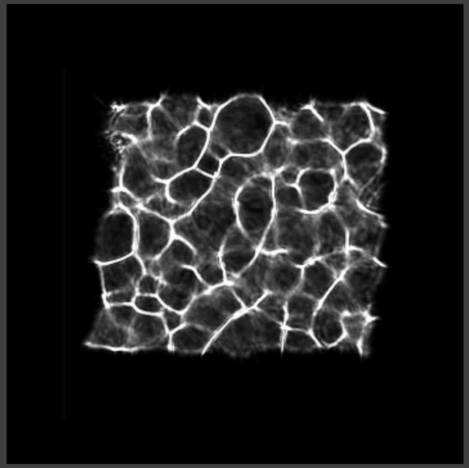




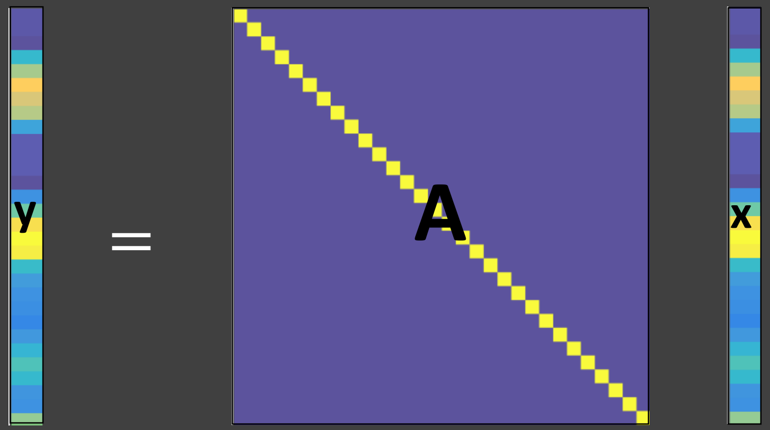
Diffuser maps points to many points (linear combination!)



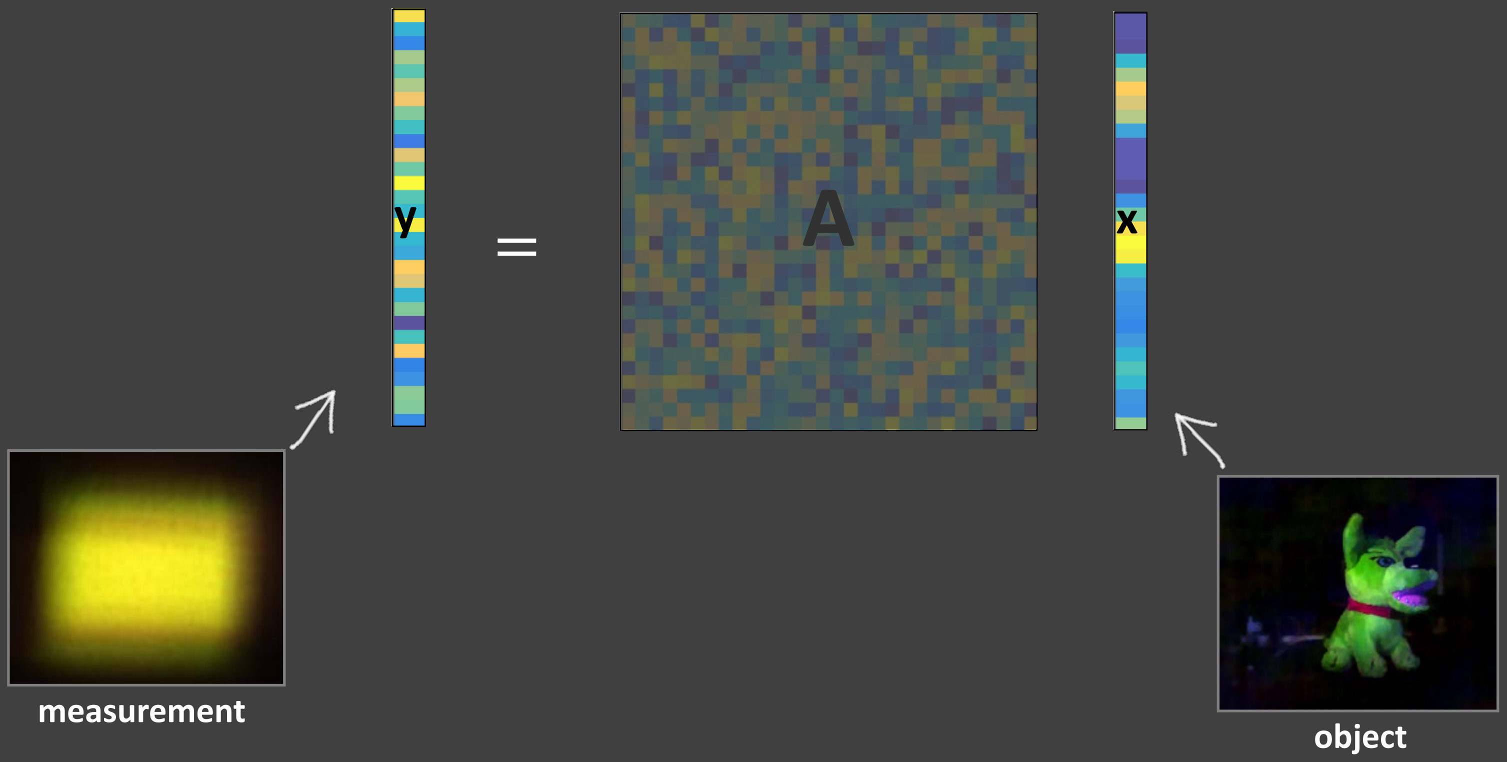
Point Spread Function (PSF)



Traditional cameras take direct measurements



Computational cameras can multiplex



raw sensor data

recovered scene

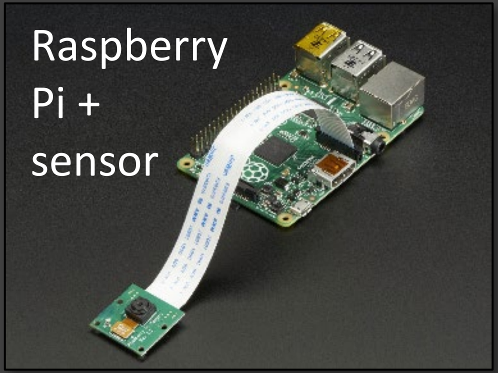

raw sensor data

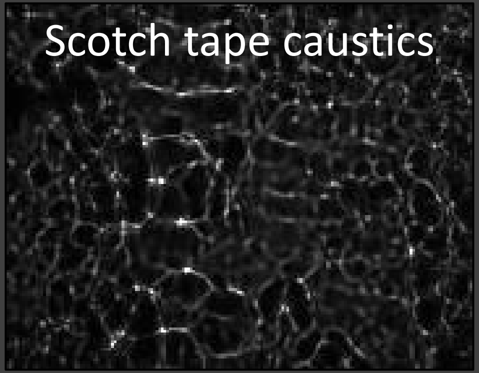

recovered scene

raw sensor data

recovered scene

El cheapo version – ScotchTapeCam!





https://waller-lab.github.io/DiffuserCam/


Camille Biscarrat
Shreyas Parthasarathy