EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 4

This homework is due February 19, 2021, at 23:59.
Self-grades are due February 22, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit each file to its respective assignment on Gradescope.

Study group task: We recommend you use some time during your study group to develop a plan for studying for the first midterm, which is coming soon. Discuss how you can support each other in your studying. Do you want to have an extra meeting to discuss some harder concepts? If you are not working with a group, you may make your plan alone. Make a list of what you want to review — this should include lectures, discussion problems, homework problems, and notes.

1. Reading Assignment

For this homework, please review Note 5 and read Notes 6, 7. The notes 5 and 6 provide an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps, and matrix inversion. Note 7 provides an introduction to vector spaces. You are always welcome and encouraged to read beyond this as well. Note 8 discusses column spaces and nullspaces, so it might be useful to read that for this homework as well.

You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?
Solution: Each column in the state transition matrix must sum to one.

2. Feedback on your study groups

Please help us understand how your study groups are going! Fill out the following survey to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

https://forms.gle/GAovoM3WxYYZYc5PA

To get full credit for this question you must (1) fill out the survey (it will record your email) and (2) indicate in your homework submission that you filled out the survey.

3. Mechanical Inverses

***Learning Objectives**: Matrices represent linear transformations, and their inverses represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

For each of the following values of matrix A:

i Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii **For parts (a)-(d)**, in addition to finding the inverse (if it exists), describe how the matrix $\mathbf{A}$ transforms an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

For example, if $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, then $\mathbf{A}$ could scale $\begin{bmatrix} x \\ y \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. If $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$, then $\mathbf{A}$ could reflect $\begin{bmatrix} x \\ y \end{bmatrix}$ across the x axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii **For parts (a)-(d)**, if we use $\mathbf{A}$ to geometrically transform $\begin{bmatrix} x \\ y \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x \\ y \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

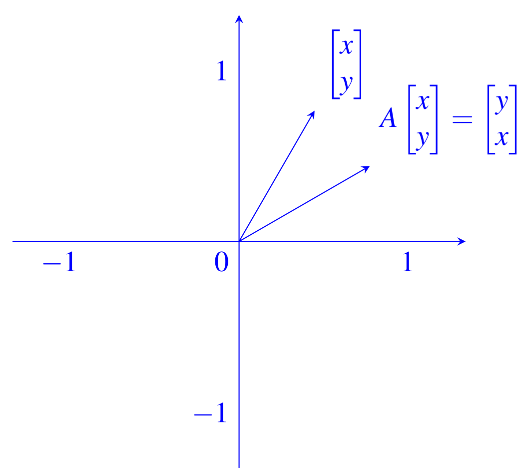
Solution:

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \text{swap } R_1, R_2$$

The inverse does exist. You can give yourself full credit if you find the inverse by applying the formula we derived in discussion 3B.

$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



The original matrix $\mathbf{A}$ flips the x and y components of the vector. Any correct equivalent sequence of operations (such as reflecting the vector across the $x = y$ line) warrants full credit. Notice how the inverse does the exact same thing–that is, it switches the x and y components of the vector it's applied to. This makes sense–switching x and y twice on a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ gives us the same vector $\begin{bmatrix} x \\ y \end{bmatrix}$. So the transformation done by $\mathbf{A}$ is reversible.

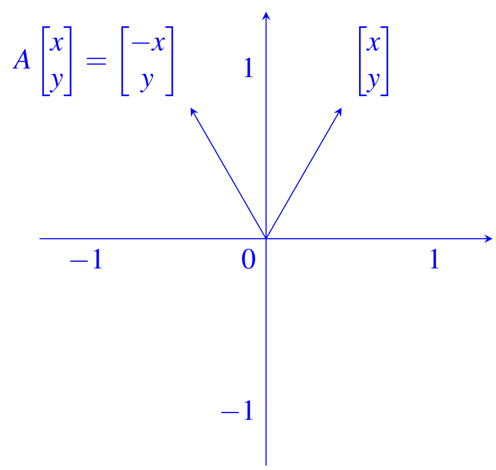
(b) $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] R_1 \leftarrow -R_1$$

The inverse does exist. You can give yourself full credit if you find the inverse by applying the formula we derived in discussion 3B.

$\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



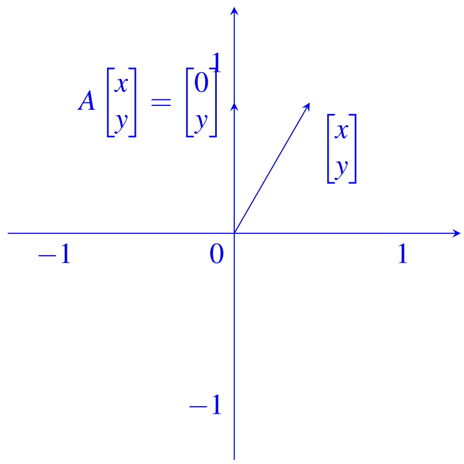
The original matrix $\mathbf{A}$ reflects the vector across the y-axis, i.e. it multiplies the vector's x-component by a factor of -1. Reflecting the vector across the y-axis again with $\mathbf{A}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ will give you the original vector, i.e. the transformation done by $\mathbf{A}$ is reversible.

(c) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{swap } \mathrm{R}_1, \mathrm{R}_2$$

We see here that the inverse does not exist because the second row represents an inconsistent equation. Another way to see that the inverse does not exist is by realizing that the first column (and first row) of the original matrix are the zero vector, so the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.



The original matrix $\mathbf{A}$ removes the x-component of the vector it's applied to and keeps the same y-component. Graphically speaking, this matrix can be thought of as taking the "shadow" of the vector on the y-axis if you were to shine a light perpendicular to the y-axis.

Since the x-component of the vector is completely lost after the transformation, the process is not reversible.

(d) $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Assume $\cos\theta \neq 0$. *Hint:* $\cos^2\theta + \sin^2\theta = 1$.

Solution:

The inverse does exist. You can give yourself full credit if you find the inverse by applying the formula we derived in discussion 3B.

$$\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1 / \cos\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \cos\theta + \frac{\sin^2\theta}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 - R_1 / \sin\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & \frac{1}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \qquad \cos^2\theta + \sin^2\theta = 1$$

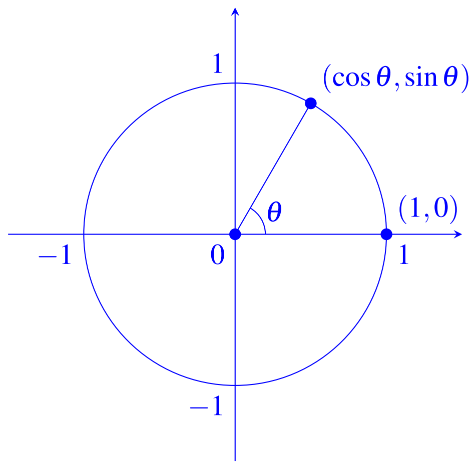
$$\rightarrow \left[\begin{array}{cc|cc} 1 & -\frac{\sin\theta}{\cos\theta} & \frac{1}{\cos\theta} & 0 \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \qquad R_2 \leftarrow R_2 \times \cos\theta$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \qquad R_1 \leftarrow R_1 - R_2 \times -\sin\theta / \cos\theta$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

The original matrix $\mathbf{A}$ is the two-dimensional rotation matrix as seen in discussion 2B. The rotation matrix rotates a vector in the counter-clockwise direction, and its inverse rotates a vector in the clockwise direction. Take the vector $\begin{bmatrix}1\\0\end{bmatrix}$ for example:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$



The inverse matrix can also be found from the rotation matrix that rotates a vector by an angle $-\theta$. The inverse matrix can also be found as follows:

$$\mathbf{A}^{-1}=\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

So the transformation done by $\mathbf{A}$ is a reversible process.

(e) $\mathbf{A}=\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}$

Solution:

We can use Gaussian elimination to find the inverse of the matrix.

$$\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 2 & 0 & 0 & 1\end{array}\right]$$

$$\rightarrow\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 0 & -2 & -2 & 1\end{array}\right] \qquad R_2 \leftarrow R_2 - 2R_1$$

$$\rightarrow\left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 0 & 1 & 1 & -\frac{1}{2}\end{array}\right] \qquad R_2 \leftarrow -R_2/2$$

$$\rightarrow\left[\begin{array}{cc|cc}1 & 0 & 0 & \frac{1}{2}\\ 0 & 1 & 1 & -\frac{1}{2}\end{array}\right] \qquad R_1 \leftarrow R_1 - R_2$$

Inverse exists: $\mathbf{A}^{-1}=\begin{bmatrix}0 & \frac{1}{2}\\ 1 & -\frac{1}{2}\end{bmatrix}$. You can give yourself full credit if you find the inverse by applying the formula we derived in discussion 3B.

(f) $\mathbf{A}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 2\\ 1 & 4 & 4\end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 4 & 4 & -1 & 0 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_2$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the second and third columns are equal, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(g) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

Solution:
We can use Gaussian elimination to find the inverse of the matrix.

$$\left[\begin{array}{ccc|ccc} -1 & 1 & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 1 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1 \times -1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \qquad R_2 \leftarrow R_2 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_2/2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \qquad R_3 \leftarrow 2R_3/3; R_2 \rightarrow R_2/2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \qquad R_2 \leftarrow R_2 + R_3/2; R_1 \leftarrow R_1 - R_3/2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \qquad R_1 \leftarrow R_1 - R_2 \times -1$$

Inverse exists: $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(h) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] R_3 \leftarrow R_3 - R_2 \times -1$$

The inverse does not exist because the last equation is inconsistent. That is, we have a row of zeros on the left hand side, corresponding to which there is no row of zeros on the right hand side. An alternative reason is that the third column is the negative of the second column, i.e., they are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

(i) **(OPTIONAL)**

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix? Hint 2: We're reasonable people!

Solution:

Inverse does not exist because $\text{column}_1 + \text{column}_2 + \text{column}_3 = \text{column}_4$, which means that the columns are linearly dependent. Since the columns of the matrix are linearly dependent, the inverse does not exist.

4. Properties of Pump Systems

Learning Objectives: *This problem illustrates how matrices and vectors can be used to represent linear transformations.*

Throughout this problem, we will consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 1, represented as a graph. Each node in the graph is marked with a letter and **represents a reservoir**. Each arrow in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The **fraction of water** is written on top of the arrow.

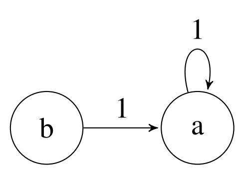


Figure 1: Pump system

(a) For the system of pumps shown in Figure 1, find the associated state transition matrix. In other words, find the matrix $\mathbf{A}$ such that:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n], \text{ where } \vec{x}[n] = \begin{bmatrix} x_a[n] \\ x_b[n] \end{bmatrix}$$

$x_a[n]$ and $x_b[n]$ represent the amount of water in reservoir a and b, respectively, at time step n.

Solution:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) Let us assume that at time step 0, the reservoirs are initialized to the following water levels: $x_a[0] = 0.5, x_b[0] = 0.5$. In a completely alternate universe, the reservoirs are initialized to the following water levels: $x_a[0] = 0.3, x_b[0] = 0.7$. For both initial states, what are the water levels at timestep 1 ($\vec{x}[1]$)? Use your answer from part (a) to compute your solution.

Solution:

$$\vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_a[0] \\ x_b[0] \end{bmatrix} = \begin{bmatrix} x_a[1] \\ x_b[1] \end{bmatrix}$$

$$\vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{x}[1] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) If you observe the reservoirs at timestep 1, i.e if you know $\vec{x}[1]$, can you figure out what the initial ($\vec{x}[0]$) water levels were? Why or why not?

Solution:

No, at timestep 1, configurations 1 and 2 are indistinguishable. If you examine the transition matrix, its columns are linearly dependent, thus there is not a unique solution to $\mathbf{A}\vec{x}[0] = \vec{x}[1]$ and it is impossible to ascertain how the water was distributed over the states at time $n = 0$ from the states at time $n = 1$.

(d) Now let us generalize what we observed. Say there is a transition matrix $\mathbf{A}$ representing a pump system, and there exist two distinct initial state vectors/water levels: $\vec{x}_u[0]$ and $\vec{x}_v[0]$, that lead to the same state vector $\vec{x}[1]$ after $\mathbf{A}$ acts on them. You do not know which of the two initial state vectors you started in. Can you decide which initial state you started in by observing $\vec{x}[1]$? What does this say about the matrix $\mathbf{A}$?

Solution:

We are told that two different initial states, $\vec{x}_u[0]$ and $\vec{x}_v[0]$, lead to the same resulting state $\vec{x}[1]$.

$$\mathbf{A}\vec{x}_u[0] = \vec{x}[1] \qquad \mathbf{A}\vec{x}_v[0] = \vec{x}[1]$$

Now, consider the system:

$$\mathbf{A}\vec{x} = \vec{x}[1].$$

We know that this system has two distinct solutions. Hence, from the theorem we proved in class we can say that the columns of $\mathbf{A}$ are linearly dependent. This implies that the matrix $\mathbf{A}$ is not invertible. Also, you cannot decide which initial state you started in by observing $\vec{x}[1]$.

(e) Now, we want to prove the following theorem in a step-by-step fashion.

Theorem: Consider a system consisting of k reservoirs such that the entries of each column in the system's state transition matrix sum to one. If s is the total amount of water in the system at timestep n, then the total amount of water at timestep $(n+1)$ will also be s.

i. Since the problem does not specify the transition matrix, let us start with a transition matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and a state vector $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$ (this is an example when $k = 2$). In general, it is helpful to write as much out mathematically as you can in proofs. It can also be helpful to draw the transition graph. Write out what is "known", i.e. ALL the information that is given to you in the theorem statement *in mathematical form*.

Solution: Each column of the transition matrix sums to one:

$$a_{11} + a_{21} = 1, \qquad a_{12} + a_{22} = 1$$

The total amount of water in the system is s at timestep n:

$$x_1[n] + x_2[n] = s$$

We know that the state vector at the next timestep is equal to the transition matrix applied to the state vector at the current timestep:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$$

ii. Now write out what is to be proved *in mathematical form*.

Solution: We want to prove that the total amount of water at timestep $(n+1)$ will also be s:

$$x_1[n+1] + x_2[n+1] = s$$

iii. Prove the statement for the case of two reservoirs.

Solution: Consider the product $\mathbf{A}\vec{x}[n] = \vec{x}[n+1]$:

$$\mathbf{A}\vec{x}[n] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} = \begin{bmatrix} a_{11}x_1[n] + a_{12}x_2[n] \\ a_{21}x_1[n] + a_{22}x_2[n] \end{bmatrix}$$

Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\sum_{i=1}^{2} x_i[n+1] = (a_{11}x_1[n] + a_{12}x_2[n]) + (a_{21}x_1[n] + a_{22}x_2[n])$$

Regrouping terms:

$$(a_{11} + a_{21})x_1[n] + (a_{12} + a_{22})x_2[n] = x_1[n] + x_2[n] = s$$

iv. Now use what you learned to generalize to the case of k reservoirs. *Hint:* Think about $\mathbf{A}$ in terms of its columns, since you have information about sum of each column.

Solution:

Let $\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_k[n] \end{bmatrix}$ be the amount of water in each reservoir at timestep n. We know:

$$x_1[n] + x_2[n] + \cdots + x_k[n] = s$$

Let $\vec{a}_j$ be the j-th column of the state transition matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix}$$

We know that every column of $\mathbf{A}$ sums to one, so we know for all j,

$$a_{1j} + a_{2j} + \cdots + a_{kj} = 1$$

Now, consider the product $\mathbf{A}\vec{x}[n]$:

$$\mathbf{A}\vec{x}[n] = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_k[n] \end{bmatrix} = x_1[n]\vec{a}_1 + x_2[n]\vec{a}_2 + \cdots + x_k[n]\vec{a}_k = \vec{x}[n+1]$$

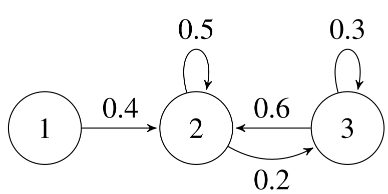
Let's consider the sum of the elements in $\vec{x}[n+1]$:

$$\begin{aligned} x_1[n+1] + x_2[n+1] + \cdots + x_k[n+1] = &(a_{11}x_1[n] + a_{12}x_2[n] + \cdots + a_{1k}x_k[n]) \\ &+ (a_{21}x_1[n] + a_{22}x_2[n] + \cdots + a_{2k}x_k[n]) \\ &+ \ldots \\ &+ (a_{k1}x_1[n] + a_{k2}x_2[n] + \cdots + a_{kk}x_k[n]) \end{aligned}$$

Factoring out each element of $x[n]$ gives:

$$\begin{aligned} &x_1[n](a_{11} + a_{21} + \cdots + a_{k1}) + x_2[n](a_{12} + a_{22} + \cdots + a_{k2}) + \cdots + x_k[n](a_{1k} + a_{2k} + \cdots + a_{kk}) \\ &= x_1[n] + x_2[n] + \cdots + x_k[n] = s \end{aligned}$$

(f) Set up the state transition matrix $\mathbf{A}$ for the system of pumps shown below. Compute the sum of the entries of each column of the state transition matrix. Are the sums greater than/less than/equal to 1? Explain what this $\mathbf{A}$ matrix physically implies about how the total amount of water in this system changes over time.
Note: If there is no "self-arrow/self-loop," you can interpret it as a self-loop with weight 0, i.e. no water returns.



Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0.4 & 0.5 & 0.6 \\ 0 & 0.2 & 0.3 \end{bmatrix}$$

Note that the entries in the columns do *not* sum to one. This is physically interpreted as a "leak" – i.e., the total amount of water is not conserved.

5. [OPTIONAL] Mechanical Basis

Learning Objectives: *Determining the basis of a vector space.*

(a) Let vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3 \in \mathbb{R}^4$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for the vector space $\mathbb{R}^4$? Justify your answer.

Solution:

No, there are only 3 vectors, but in order to span $\mathbb{R}^4$, you need at least 4 vectors so that every vector in $\mathbb{R}^4$ space can be formed by a linear combination of the basis vectors.

(b) Let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}$. Given a new set of vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be a basis for $\mathbb{R}^3$? If so, express $\vec{x}$ as a linear combination of these basis vectors.

If the set of five vectors cannot form a basis for $\mathbb{R}^3$, choose a new basis including $\vec{v}_1$, $\vec{v}_2$ and any number of additional vectors from the set. Then express $\vec{x}$ as a linear combination of the newly constructed basis vectors.

Solution:

The set of five vectors is NOT a basis, since there are 5 vectors for a 3 dimensional space and a basis is defined as the set of minimum number of vectors required to span the vector space.

Since we are told to use $\vec{v}_1$ and $\vec{v}_2$ in the basis, we only need to choose the appropriate third vector: a vector that is linearly independent of $\vec{v}_1$ and $\vec{v}_2$. By inspection, we see that

$$\vec{v}_3 = 3\vec{v}_1$$

$$\vec{v}_5 = \vec{v}_1 + \vec{v}_2$$

Therefore, $\vec{v}_4$ is the only vector that is linearly independent of $\vec{v}_1$ and $\vec{v}_2$. So $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ form the basis. Hence the $\vec{x}$ can be expressed as:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_4\vec{v}_4$$

We can use Gaussian elimination to determine the coefficients, c_1, c_2, and c_3:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 2 & 6 \end{array}\right] \qquad R_2 \leftarrow R_2 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 4 & 8 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_3 \leftarrow R_3/4$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_1 \leftarrow R_1 - 2R_3; R_2 \leftarrow R_2 + 2R_3$$

Therefore, we get the linear combination

$$\vec{x} = \vec{v}_1 + 2\vec{v}_2 + 2\vec{v}_4.$$

(c) Let $\vec{x} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$. Given a new set of vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be a basis for $\mathbb{R}^3$? If so, express $\vec{x}$ as a linear combination of these basis vectors.

If the set of five vectors cannot form a basis for $\mathbb{R}^3$, choose a new basis including $\vec{v}_1$, $\vec{v}_2$ and any number of additional vectors from the set. Then express $\vec{x}$ as a linear combination of the newly constructed basis vectors.

Solution:

The set of five vectors is NOT a basis, since there are 5 vectors for a 3 dimensional space and a basis is defined as the set of minimum number of vectors required to span the vector space.

Since we are told to use $\vec{v}_1$ and $\vec{v}_2$ in the basis, we only need to choose the appropriate third vector: a vector that is linearly independent of $\vec{v}_1$ and $\vec{v}_2$. By inspection, we see that

$$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$$

$$\vec{v}_5 = \vec{v}_1 + 3\vec{v}_2$$

Therefore, $\vec{v}_4$ is the only vector that is linearly independent of $\vec{v}_1$ and $\vec{v}_2$. So $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ form the basis. Hence the $\vec{x}$ can be expressed as:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_4\vec{v}_4$$

We can use Gaussian elimination to determine the coefficients, c_1, c_2, and c_3:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 3 & 1 & 0 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -6 & -4 \\ 0 & 1 & 2 & 4 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, we get the linear combination

$$\vec{x} = \vec{v}_1 + 2\vec{v}_2 + \vec{v}_4$$

6. Finding Null Spaces and Column Spaces

Learning Objectives: *Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as $\mathrm{N}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{N}(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the set of $\mathbf{A}$'s columns. The column space can be notated as $\mathrm{C}(\mathbf{A})$ or $\mathrm{range}(\mathbf{A})$ and the definition can be written in set notation as:

$$\mathrm{C}(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors - i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3\times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $\mathrm{C}(\mathbf{A})$?

Solution: If you are stuck solving a problem like this, consider concrete examples. We want to find the maximum possible number of linearly independent column vectors, so we look for examples and check if we can exceed certain values.

Consider the following example matrix, where the entries marked with $*$ are arbitrary values:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$$

Here all 5 columns are $\in \mathbb{R}^3$. The first three columns are linearly independent, so at least three linearly independent columns are achievable. The first three columns span $\mathbb{R}^3$, therefore any choice of fourth and fifth columns, also in $\mathbb{R}^3$, can be written as a linear combination of the first three columns. This means that we cannot exceed three linearly independent columns. Thus the maximum number of linearly independent column vectors is 3. In general, if $m < n$, then the columns of $\mathbf{A} \in \mathbb{R}^{m\times n}$ will always be linearly dependent, since you cannot have more than m linearly independent columns in $\mathbb{R}^m$.

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\mathrm{C}(\mathbf{A})$ (i.e. a basis for $\mathrm{C}(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $\mathrm{C}(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

Solution: $\mathrm{C}(\mathbf{A})$ is the space spanned by its columns, so the set of all columns is a valid span for $\mathrm{C}(\mathbf{A})$. However, we are asking you to choose a subset of the columns and still span $\mathrm{C}(\mathbf{A})$, as we showed in part (a). To find the minimum number of columns needed and determine the dimension of $\mathrm{C}(\mathbf{A})$, we can remove vectors from the set of columns until we are left with a linearly independent set.

By inspection, the second, fourth, and fifth columns can be omitted from a set of columns as they can be expressed as linear combinations of the first and third columns. Thus the dimension of $\mathbf{A}$ is 2.

One set spanning $\mathrm{C}(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Another valid set of vectors which span $\mathrm{C}(\mathbf{A})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Note with this second set, none of the columns of $\mathbf{A}$ appear. Despite this, the span of this set will still be equal to $\mathrm{C}(\mathbf{A})$, which for this matrix is the set of all vectors in $\mathbb{R}^3$ with zero third entry. Geometrically, both of these solutions span the same plane, i.e. the xy-plane in the 3D space.

Give yourself full credit if you recognized that the dimension was 2, and if you had a *minimum* set of vectors that spans $\mathrm{C}(\mathbf{A})$.

(c) Find a *minimum* set of vectors that span $\mathrm{N}(\mathbf{A})$ (i.e. a basis for $\mathrm{N}(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $\mathrm{N}(\mathbf{A})$?

Solution:

Finding $\mathrm{N}(\mathbf{A})$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + x_2 - 2x_4 + 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

We observe that x_2, x_4, and x_5 are free variables, since they correspond to the columns with no pivots. Thus, we let $x_2 = a$, $x_4 = b$, and $x_5 = c$. Now we rewrite the equations as:

$$\begin{aligned} x_1 &= -a + 2b - 3c \\ x_2 &= a \\ x_3 &= b - c \\ x_4 &= b \\ x_5 &= c \end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\mathrm{N}(\mathbf{A})$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of $\mathrm{N}(\mathbf{A})$ is 3, as it is the minimum number of vectors we need to span it.

(d) Find the sum of the dimensions of $\mathrm{N}(\mathbf{A})$ and $\mathrm{C}(\mathbf{A})$. What do you notice about this sum in relation to the dimensions of $\mathbf{A}$?

Solution: The dimensions of $\mathrm{C}(\mathbf{A})$ and $\mathrm{N}(\mathbf{A})$ add up to the number of columns in $\mathbf{A}$. This is true of all matrices. This relates to what is known as the rank-nullity theorem; however we will not be covering this in 16A. You'll get to explore this in 16B.

(e) Now consider the new matrix, $\mathbf{B} = \mathbf{A}^T$,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $\mathrm{C}(\mathbf{B})$ (i.e. a basis for $\mathrm{C}(\mathbf{B})$). What is the minimum number of vectors required to span the $\mathrm{C}(\mathbf{B})$?

Solution:

We see that the first two column vectors of $\mathbf{B}$ are linearly independent and sufficient to span $\mathrm{C}(\mathbf{B})$, since the third column is trivial (all zeros) and does not contribute anything to the span. Therefore, $\mathrm{C}(\mathbf{B})$ has dimension 2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(f) You are given the following matrix $\mathbf{G}$. Find a *minimum* set of vectors that span $\mathrm{N}(\mathbf{G})$, i.e. a basis for $\mathrm{N}(\mathbf{G})$.

$$\mathbf{G} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

Solution: To find $\mathrm{N}(\mathbf{G})$, we wish to solve for all $\vec{x}$ such that $\mathbf{G}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 2 & -4 & 4 & 8 & 0 \\ 1 & -2 & 3 & 6 & 0 \\ 2 & -4 & 5 & 10 & 0 \\ 3 & -6 & 7 & 14 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 1 & -2 & 3 & 6 & 0 \\ 2 & -4 & 5 & 10 & 0 \\ 3 & -6 & 7 & 14 & 0 \end{array}\right] \qquad R_1 \leftarrow \frac{1}{2}R_1$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \qquad \begin{array}{c} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 - 3R_1 \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{c} R_1 \leftarrow R_1 - 2R_2 \\ R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2 \end{array}$$

Vectors in $\mathrm{N}(\mathbf{G})$ satisfy the following equations:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} \implies \begin{aligned} x_1 - 2x_2 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

We then assign free variables $x_2 = a$ and $x_4 = b$ and substitute in:

$$\begin{aligned} x_1 &= 2a \\ x_2 &= a \\ x_3 &= -2b \\ x_4 &= b \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, $\mathrm{N}(\mathbf{G})$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(g) (**OPTIONAL**) For the following matrix $\mathbf{D}$, find $\mathrm{C}(\mathbf{D})$ and its dimension, and $\mathrm{N}(\mathbf{D})$ and its dimension.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

Solution:

To find $\mathrm{C}(\mathbf{D})$, we identify the linearly independent columns of $\mathbf{D}$ by inspection. The second column is a scaled version of the first column. The third column is linearly independent from the first and second columns, since it is not a scaled version of the first column. Finally, the fourth column is simply the first column minus the third column and thus is linearly dependent with respect to prior columns.

So we conclude that the linearly independent columns of $\mathbf{D}$ are the first and third columns so that a basis for $\mathrm{C}(\mathbf{D})$ is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -1 \end{bmatrix} \right\}$$

and thus the dimension of $\mathrm{C}(\mathbf{D})$ is 2.

To find $\mathrm{N}(\mathbf{D})$, we can row reduce the matrix to find solutions to $\mathbf{D}\vec{x} = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & -1 & -3 & 4 & 0 \\ 3 & -3 & -5 & 8 & 0 \\ 1 & -1 & -1 & 2 & 0 \end{array}\right] \xrightarrow[\text{rref}]{} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since we only have pivots in the first and third columns, we can assign the free variables $x_2 = s$ and $x_4 = t$. We can write all solutions to $\mathbf{D}\vec{x} = \vec{0}$ as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s-t \\ s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} t$$

A basis for $\mathrm{N}(\mathbf{D})$ is:

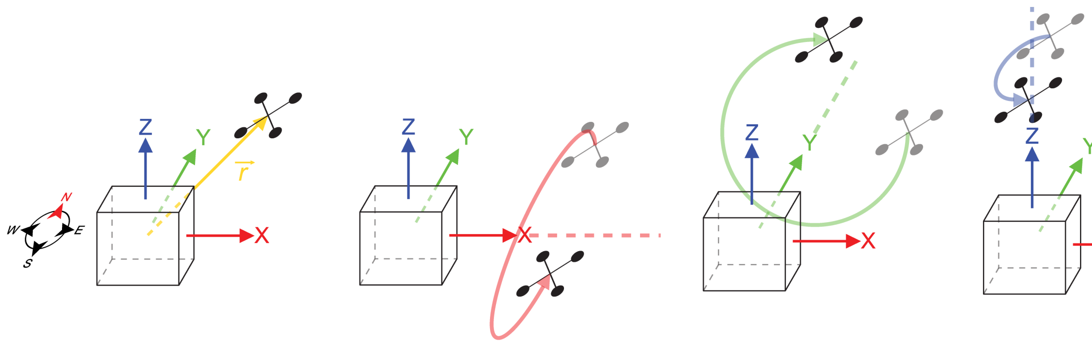
$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and thus the dimension of $\mathrm{N}(\mathbf{D})$ is 2.

7. [OPTIONAL] Quadcopter Transformations

Learning Objectives: *Linear algebra is often used to represent transformations in robotics. This problem introduces some of the basic uses of transformations.*

Vijay and his colleagues are interested in developing a communication link using a laser to control the location of a quadrotor. Consider a vector $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ representing the location of the quadcopter relative to the origin. The quadcopter is only capable of three different maneuvers relative to the origin. The



maneuvers are rotations about the x, y, and z axes. For perspective, the positive x-axis points east, the positive y-axis points north, and the positive z-axis points up towards the sky. The figures below illustrate the quadcopter and these maneuvers.

We can represent each of these rotations, that are linear transformations, as matrices that operate on the location vector of the quadcopter, $\vec{r}$, to position it at its new location. The matrices $\mathbf{R_x}(\theta)$, $\mathbf{R_y}(\psi)$, and $\mathbf{R_z}(\phi)$ represent rotations about the x-axis, y-axis, and z-axis, respectively. The matrices are:

$$\mathbf{R_x}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \mathbf{R_y}(\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}, \mathbf{R_z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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(a) Vijay wants to make the quadcopter rotate first by 30° about the x-axis, and then by 60° about the z-axis. Use $\mathbf{R_x}(\theta)$, $\mathbf{R_y}(\psi)$, and $\mathbf{R_z}(\phi)$ to construct **a single matrix that performs the operations in the specified order**. Show the matrix operations and calculations by hand.

Solution: First, we must apply $\mathbf{R_x}(\theta)$ to $\vec{r}$, then $\mathbf{R_z}(\phi)$. We are told the angle of rotation about x-axis, θ, is 30°. Likewise, for the z-axis, $\phi = 60°$. So the matrix is:

$$\mathbf{R_z}(60°)\,\mathbf{R_x}(30°) = \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\cdot 1 + 0 + 0 & 0 - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + 0 & 0 + -\frac{\sqrt{3}}{2}\cdot(-\frac{1}{2}) + 0 \\ \frac{\sqrt{3}}{2}\cdot 1 + 0 + 0 & 0 + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + 0 & 0 + \frac{1}{2}\cdot(-\frac{1}{2}) + 0 \\ 0 + 0 + 0 & 0 + 0 + 1\cdot\frac{1}{2} & 0 + 0 + 1\cdot\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(b) Vijay accidentally punched in the two rotation commands in reverse. The 60° rotation about the z-axis occurred before the 30° rotation about the x-axis. Use $\mathbf{R_x}(\theta)$, $\mathbf{R_y}(\psi)$, and $\mathbf{R_z}(\phi)$ to construct **a single matrix that performs the operations in the accidentally reversed order**. Show the matrix operations and calculations by hand.

Solution: Now, we compute the matrix-matrix product with the order of $\mathbf{R_z}(\phi)$ and $\mathbf{R_x}(\theta)$ reversed.

$$\mathbf{R_x}(30°)\,\mathbf{R_z}(60°) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} \cos 60° & -\sin 60° & 0 \\ \sin 60° & \cos 60° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot\frac{1}{2}+0+0 & 1\cdot(-\frac{\sqrt{3}}{2})+0+0 & 0+0+0 \\ 0+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}+0 & 0+\frac{\sqrt{3}}{2}\cdot\frac{1}{2}+0 & 0+0-\frac{1}{2}\cdot 1 \\ 0+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+0 & 0+\frac{1}{2}\cdot\frac{1}{2}+0 & 0+0+\frac{\sqrt{3}}{2}\cdot 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(c) Say the quadcopter was initially positioned at $\vec{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

i. Where did Vijay intend for the quadcopter to end up? Use your result from part (a) to find this out.

ii. Where did the quadcopter actually end up with the accidentally reversed order of the rotations? Use your result from part (b) to find this out.

iii. Did the quadcopter end up where it was supposed to go?

Solution: The intended location is given by:

$$\mathbf{R_z}(60^\circ)\,\mathbf{R_x}(30^\circ)\,\vec{r} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-1+2\sqrt{3}}{4} \\ \frac{-2+3\sqrt{3}}{4} \\ \frac{1+2\sqrt{3}}{2} \end{bmatrix}$$

The actual location is given by:

$$\mathbf{R_x}(30^\circ)\,\mathbf{R_z}(60^\circ)\,\vec{r} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{-1+\sqrt{3}}{4} \\ \frac{1+5\sqrt{3}}{4} \end{bmatrix}$$

We can see that $\begin{bmatrix} \frac{-1+2\sqrt{3}}{4} \\ \frac{-2+3\sqrt{3}}{4} \\ \frac{1+2\sqrt{3}}{2} \end{bmatrix} \neq \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{-1+\sqrt{3}}{4} \\ \frac{1+5\sqrt{3}}{4} \end{bmatrix}$, and that they are not the same.

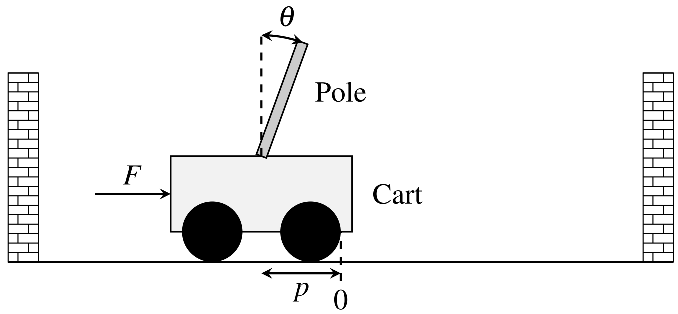
8. Segway Tours

Learning Objective: *The learning objective of this problem is to see how the concept of span can be applied to control problems. If a desired state vector of a linear control problem is in a span of a particular set of vectors, then the system may be steered to reach that particular vector using the available inputs.*

Your friends have decided to start a new SF tour business, and you suggest they use segways. They become intrigued by your idea and asks you how a segway works. A segway is essentially a stand on two wheels.

The segway works by applying a force (through the spinning wheels) to the base of the segway. This controls both the position on the segway and the angle of the stand. As the driver pushes on the stand, the segway tries to bring itself back to the upright position, and it can only do this by moving the base.

Is it possible for the segway to be brought upright and to a stop from any initial configuration? There is only one input (force) used to control two outputs (position and angle). You talk to a friend who is GSIing EE128, and she tells you that a segway can be modeled as a cart-pole system.



A cart-pole system can be fully described by its position p, velocity $\dot{p}$, angle θ, and angular velocity $\dot{\theta}$. We write this as a "state vector", $\vec{x}$:

$$\vec{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}.$$

The input to this system is a scalar quantity $u[n]$ at time n, which is the force F applied to the cart (or base of the segway).[1]

The cart-pole system can be represented by a linear model:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n], \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{4\times4}$ and $\vec{b} \in \mathbb{R}^{4\times1}$.

The control $u[n]$ allows us to move the state ($\vec{x}$) in the direction of $\vec{b}$. So, if $u[n] = 2$, we move the state by $2\vec{b}$ at time n, and so on. We can choose different controls at different times.

The model tells us how the state vector, $\vec{x}$, will evolve over time as a function of the current state vector and control inputs.

You look at this general linear system and try to answer the following question: Starting from some initial state $\vec{x}_0$, can we reach a final desired state, $\vec{x}_f$, in N steps?

The challenge seems to be that the state is four-dimensional and keeps evolving and that we can only apply a one-dimensional (scalar) control at each time. Typically, to set the values of four variables to desired quantities, you would need four inputs. Can you do this with just one input?

We will solve this problem by walking through several steps.

(a) Express $\vec{x}[1]$ in terms of $\vec{x}[0]$ and the input $u[0]$.

Solution:

From Equation (1), we get (by substituting $n = 0$):

$$\vec{x}[1] = \mathbf{A}\vec{x}[0] + \vec{b}u[0] \tag{2}$$

[1] You might note that velocity and angular velocity are derivatives of position and angle respectively. Differential equations are used to describe continuous time systems, which you will learn more about in EECS 16B. But even without these techniques, we can still approximate the solution to be a continuous time system by modeling it as a discrete time system where we take very small steps in time. We think about applying a force constantly for a given finite duration and we see how the system responds after that finite duration.

(b) i. Express $\vec{x}[2]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$ and $u[1]$.
ii. Then express $\vec{x}[3]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, and $u[2]$.
iii. Finally express $\vec{x}[4]$ in terms of *only* $\vec{x}[0]$ and the inputs, $u[0]$, $u[1]$, $u[2]$, and $u[3]$.
Your expressions can have other relevant variables (e.g. A, $\vec{b}$ etc) and mathematical operators.

Solution:

From Equation (1), we get (by substituting $n = 1$):

$$\vec{x}[2] = \mathbf{A}\vec{x}[1] + \vec{b}u[1]$$

By substituting $\vec{x}[1]$ from Equation (2), we get

$$\begin{aligned}\vec{x}[2] &= \mathbf{A}\vec{x}[1] + \vec{b}u[1] \\ &= \mathbf{A}\left(\mathbf{A}\vec{x}[0] + \vec{b}u[0]\right) + \vec{b}u[1] \\ &= \mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1] \end{aligned} \tag{3}$$

From Equation (1), we get (by substituting $n = 2$):

$$\vec{x}[3] = \mathbf{A}\vec{x}[2] + \vec{b}u[2]$$

By substituting $\vec{x}[2]$ from Equation (3), we get

$$\begin{aligned}\vec{x}[3] &= \mathbf{A}\vec{x}[2] + \vec{b}u[2] \\ &= \mathbf{A}\left(\mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1]\right) + \vec{b}u[2] \\ &= \mathbf{A}^3\vec{x}[0] + \mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2] \end{aligned} \tag{4}$$

From Equation (1), we get (by substituting $n = 3$):

$$\vec{x}[4] = \mathbf{A}\vec{x}[3] + \vec{b}u[3]$$

By substituting $\vec{x}[3]$ from Equation (4), we get

$$\begin{aligned}\vec{x}[4] &= \mathbf{A}\vec{x}[3] + \vec{b}u[3] \\ &= \mathbf{A}\left(\mathbf{A}^3\vec{x}[0] + \mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2]\right) + \vec{b}u[3] \\ &= \mathbf{A}^4\vec{x}[0] + \mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] \end{aligned} \tag{5}$$

(c) Now, generalize the pattern you saw in the earlier part to write an expression for $\vec{x}[N]$ in terms of $\vec{x}[0]$ and the inputs from $u[0], \ldots, u[N-1]$. **Your expression can have other relevant variables (e.g. A, $\vec{b}$ etc) and mathematical operators.**

Solution:

Use the same procedure as above for N steps. You will obtain the following expression:

$$\vec{x}[N] = \mathbf{A}^N\vec{x}[0] + \mathbf{A}^{N-1}\vec{b}u[0] + \ldots + \mathbf{A}\vec{b}u[N-2] + \vec{b}u[N-1] \tag{6}$$

You might also use the compact expression:

$$\vec{x}[N] = \mathbf{A}^N\vec{x}[0] + \sum_{i=0}^{N-1}\mathbf{A}^i\vec{b}u[N-i-1] \tag{7}$$

Note that $\mathbf{A}^0$ is the identity matrix.

As a sanity check, plug the values $N = 1, 2, 3,$ and 4 to obtain Equations (2), (3), (4), and (5), respectively.

For the next four parts of the problem, you are given the matrix $\mathbf{A}$ and the vector $\vec{b}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.05 & -0.01 & 0 \\ 0 & 0.22 & -0.17 & -0.01 \\ 0 & 0.10 & 1.14 & 0.10 \\ 0 & 1.66 & 2.85 & 1.14 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0.01 \\ 0.21 \\ -0.03 \\ -0.44 \end{bmatrix}$$

Assume the cart-pole starts in an initial state $\vec{x}[0] = \begin{bmatrix} -0.3853493 \\ 6.1032227 \\ 0.8120005 \\ -14 \end{bmatrix}$, and you want to reach the desired state $\vec{x}_f = \vec{0}$ using the control inputs $u[0], u[1], \ldots$ etc. The state vector $\vec{x}_f = \vec{0}$ corresponds to the cart-pole (or segway) being upright and stopped at the origin. **Reaching $\vec{x}_f = \vec{0}$ in N steps means that, given that we start at $\vec{x}[0]$, we can find control inputs ($u[0], u[1], \ldots$ etc), such that we get $\vec{x}[N]$ (i.e. state vector at Nth time step) equal to $\vec{x}_f = \vec{0}$.**

Note: Please use the Jupyter notebook to solve parts (d) - (g) of the problem. You may use the function we provided `gauss_elim(matrix)` to help you find the **upper triangular form** of matrices. An example of Gaussian Elimination using (`gauss_elim(matrix)`) is provided in the Jupyter notebook under section **Example Usage of gauss_elim**. You may also use the function (`np.linalg.solve`) to solve the equations.

(d) Can you reach $\vec{x}_f$ in *two* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations.

(*Hint: Express $\vec{x}[2] - \mathbf{A}^2\vec{x}[0]$ in terms of the inputs $u[0]$ and $u[1]$. Then determine if the system of equations can be solved to obtain $u[0]$ and $u[1]$. If we obtain valid solutions for $u[0]$ and $u[1]$, then we can say we will reach $\vec{x}_f$ in two time steps. Use the notebook to see if the system of equations can be solved.*)

Solution:

No.

From Equation (3), we know that $\mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}[2]$ which is equivalent to $\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}[2] - \mathbf{A}^2\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in two time steps (that is, $\vec{x}[2] = \vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = \vec{x}_f - \mathbf{A}^2\vec{x}[0],$$

where $u[0]$ and $u[1]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}\vec{b}u[0] + \vec{b}u[1] = -\mathbf{A}^2\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix} | & | \\ \mathbf{A}\vec{b} & \vec{b} \\ | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix} = -\mathbf{A}^2\vec{x}[0],$$

which yields the following augmented matrix:

$$\left[\begin{array}{cc|c} | & | & | \\ \mathbf{A}\vec{b} & \vec{b} & -\mathbf{A}^2\vec{x}[0] \\ | & | & | \end{array}\right].$$

By plugging in the values of $\mathbf{A}$, $\vec{b}$, and $\vec{x}[0]$, we get the following augmented matrix:

$$\left[\begin{array}{cc|c} 0.0208 & 0.01 & 0.02243475295 \\ 0.0557 & 0.21 & -0.30785116611 \\ -0.0572 & -0.03 & 0.0619347608 \\ -0.2385 & -0.44 & 1.38671325508 \end{array}\right].$$

Applying Gaussian elimination, we get the upper triangular form to be

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right],$$

which means that the system is inconsistent (due to the third row) and that there are no solutions for $u[0]$ and $u[1]$. It is fine if you did not row reduce all the way to the upper triangular form as long as you showed that the system of equations is inconsistent.

(e) Can you reach $\vec{x}_f$ in *three* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations.
(*Hint: Similar to the last part, express* $\vec{x}[3] - \mathbf{A}^3\vec{x}[0]$ *in terms of the inputs* $u[0]$, $u[1]$ *and* $u[2]$. *Then determine if we can obtain valid solutions for* $u[0]$, $u[1]$ *and* $u[2]$.)

Solution:

No.

Similar to the previous part, from Equation (4), we know that $\mathbf{A}^3\vec{x}[0] + \mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2] = \vec{x}[3]$, which is equivalent to $\mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2] = \vec{x}[3] - \mathbf{A}^3\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in three time steps (that is, $\vec{x}[3] = \vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2] = \vec{x}_f - \mathbf{A}^3\vec{x}[0],$$

where $u[0], u[1]$, and $u[2]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}^2\vec{b}u[0] + \mathbf{A}\vec{b}u[1] + \vec{b}u[2] = -\mathbf{A}^3\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix} | & | & | \\ \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \end{bmatrix} = -\mathbf{A}^3\vec{x}[0],$$

which yields the following augmented matrix:

$$\left[\begin{array}{ccc|c} | & | & | & | \\ \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} & -\mathbf{A}^3\vec{x}[0] \\ | & | & | & | \end{array}\right].$$

By plugging in the values of $\mathbf{A}$, $\vec{b}$, and $\vec{x}[0]$, we get the following augmented matrix:

$$\left[\begin{array}{ccc|c} 0.024157 & 0.0208 & 0.01 & 0.0064228470365 \\ 0.024363 & 0.0557 & 0.21 & -0.092123298431 \\ -0.083488 & -0.0572 & -0.03 & 0.178491836209001 \\ -0.342448 & -0.2385 & -0.44 & 1.246334243328597 \end{array}\right].$$

Applying Gaussian elimination, we get the upper triangular form to be

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

which means that the system is inconsistent (due to the fourth row) and that there are no solutions for $u[0], u[1]$, and $u[2]$. It is fine if you did not row reduce all the way to the upper triangular form as long as you showed that the system of equations is inconsistent.

(f) Can you reach $\vec{x}_f$ in *four* time steps? Show work to justify your answer. You should manipulate the equations on paper, but then use the Jupyter notebook for numerical computations. (*Use the hints from the last two parts.*)

Solution:

Yes.

Similar to the previous part, from Equation (5), we know that $\mathbf{A}^4\vec{x}[0] + \mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}[4]$ which is equivalent to $\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}[4] - \mathbf{A}^4\vec{x}[0]$.

This means that in order to reach any state $\vec{x}_f$ in four time steps (that is, $\vec{x}[4] = \vec{x}_f$), we have to solve the following system of linear equations:

$$\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = \vec{x}_f - \mathbf{A}^4\vec{x}[0],$$

where $u[0], u[1], u[2]$, and $u[3]$ are the unknowns.

Since in our case we want to reach $\vec{x}_f = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, the system of linear equations simplifies to

$$\mathbf{A}^3\vec{b}u[0] + \mathbf{A}^2\vec{b}u[1] + \mathbf{A}\vec{b}u[2] + \vec{b}u[3] = -\mathbf{A}^4\vec{x}[0].$$

In matrix form, this system of linear equations is

$$\begin{bmatrix} | & | & | & | \\ \mathbf{A}^3\vec{b} & \mathbf{A}^2\vec{b} & \mathbf{A}\vec{b} & \vec{b} \\ | & | & | & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix} = -\mathbf{A}^4\vec{x}[0].$$

Refer to the code in the solution IPython notebook for the values of the augmented matrix. After performing Gaussian elimination, we get the upper triangular form to be

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -13.24875075 \\ 0 & 1 & 0 & 0 & 23.73325125 \\ 0 & 0 & 1 & 0 & -11.57181872 \\ 0 & 0 & 0 & 1 & 1.46515973 \end{array}\right].$$

This indicates that there exists a unique solution to the system of equations, which is

$$\begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix} = \begin{bmatrix} -13.24875075 \\ 23.73325125 \\ -11.57181872 \\ 1.46515973 \end{bmatrix},$$

(g) If you have found that you can get to the final state in 4 time steps, find the required correct control inputs, i.e. $u[0]$, $u[1]$, $u[2]$ and $u[3]$, using Jupyter and verify the answer by entering these control inputs into the **Plug in your controller** section of the code in the Jupyter notebook. You need to just show that you reached the desired final state $\vec{x}_f$ by plugging in the control inputs. The code has been already written to simulate this system.
Suggestion: See what happens if you enter all four control inputs equal to 0. This gives you an idea of how the system naturally evolves!
Solution:
See the solution to the previous part.

(h) Let us reflect on what we just did. Recall the system we have:

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n] + \vec{b}u[n].$$

The control allows us to move the state at time step $n+1$ by $u[n]$ in direction $\vec{b}$, remember $u[n] \in \mathbb{R}$ is just a scalar. We know from part (c) that:

$$\vec{x}[2] = \mathbf{A}^2\vec{x}[0] + \mathbf{A}\vec{b}u[0] + \vec{b}u[1].$$

Again, here $u[0], u[1] \in \mathbb{R}$ can be thought of as arbitrary scalars, and $\mathbf{A}\vec{b}u[0] + \vec{b}u[1]$ can be thought of as the set of all linear combinations of the vectors $\vec{b}$ and $\mathbf{A}\vec{b}$. Using this observation, can you express the possible states you can arrive at in two time steps using the span of exactly *two* vectors plus a vector offset?
Solution: You can move in directions $\vec{b}$ and $\mathbf{A}\vec{b}$. We also need to account for the zero input behavior of the system reflected by the term $\mathbf{A}^2\vec{x}[0]$. Hence you can reach all positions that are in

$$\text{span}\left\{\vec{b}, \mathbf{A}\vec{b}\right\} + \mathbf{A}^2\vec{x}[0].$$

(i) Let's try to generalize the idea in the previous part. Express the states you can reach in N timesteps as a span of some vectors plus a vector offset. (Hint: Consider the direction that each control input $u[0],\ldots,u[N-1]$ can move $\vec{x}[N]$ by.)

Solution: In N timesteps, the control inputs $u[0],\ldots,u[N-1]$ allow you to move the state $\vec{x}[N]$ by any vector in

$$\text{span}\left\{\vec{b},\mathbf{A}\vec{b},\mathbf{A}^2\vec{b},\ldots,\mathbf{A}^{N-1}\vec{b}\right\}.$$

You will also have an offset of $\mathbf{A}^N\vec{x}[0]$, which captures the zero input system behavior. Taking this into consideration you can reach all states that are in

$$\text{span}\left\{\vec{b},\mathbf{A}\vec{b},\mathbf{A}^2\vec{b},\ldots,\mathbf{A}^{N-1}\vec{b}\right\}+\mathbf{A}^N\vec{x}[0].$$

(j) **(OPTIONAL)** Now say you wanted to reach anywhere in $\mathbb{R}^4$, i.e. $\vec{x}_f$ is an unspecified vector in $\mathbb{R}^4$. Under what conditions can you guarantee that you can "reach" $\vec{x}_f$ from any $\vec{x}_0$ in N time steps? Wouldn't this be cool?

Solution: This builds on the previous parts. Since now you want to reach anywhere in $\mathbb{R}^4$, and you know the possible states you can reach are given by

$$\text{span}\left\{\vec{b},\mathbf{A}\vec{b},\mathbf{A}^2\vec{b},\ldots,\mathbf{A}^{N-1}\vec{b}\right\},$$

you need this span to be $\mathbb{R}^4$.

P.S.: Congratulations! You have just derived the condition for "controllability" for systems with linear dynamics. When dealing with a system that evolves over time, we can sometimes influence the behavior of the system through various control inputs (for example, the steering wheel and gas pedal of a car or the rudder of an airplane). It is of great importance to know what states (think positions and velocities of a car or configurations of an aircraft) that our system can be controlled to. Controllability is the ability to control the system to any possible state or configuration.

A more detailed argument follows, but the above argument is sufficient to conclude the result.

More detailed argument:

The key step here is to rewrite the equation you derived in part ((c)) (Equation (7)) as

$$\sum_{i=0}^{N-1}\mathbf{A}^i\vec{b}u[N-i-1]=\vec{x}[N]-\mathbf{A}^Nx[0].$$

$\vec{x}[N]=\vec{x}_f$ can be anything in $\mathbb{R}^4$. Therefore, the system of linear equations can be written as

$$\sum_{i=0}^{N-1}\mathbf{A}^i\vec{b}u[N-i-1]=\vec{x}_f-\mathbf{A}^Nx[0].$$

If we extend this sum, we get

$$\mathbf{A}^{N-1}\vec{b}u[0]+\mathbf{A}^{N-2}\vec{b}u[1]+\cdots+\mathbf{A}\vec{b}u[N-2]+\vec{b}u[N-1]=\vec{x}_f-\mathbf{A}^Nx[0].$$

This system of linear equations can be further rewritten as

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}\begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N-1] \end{bmatrix}=\vec{x}_f-\mathbf{A}^Nx[0].$$

For this system to be solvable, we need $\vec{x}_f - \mathbf{A}^N x[0] \in \text{span}\left\{\vec{b}, \mathbf{A}\vec{b}, \mathbf{A}^2\vec{b}, \ldots, \mathbf{A}^{N-1}\vec{b}\right\}$. Since $\vec{x}_f$ can be any vector in $\mathbb{R}^4$, it also means that $\vec{x}_f - \mathbf{A}^N x[0]$ can be any vector in $\mathbb{R}^4$. This means that in order to be able to reach any state $\vec{x}_f \in \mathbb{R}^4$, the range (column space) of the matrix $\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{A}^{N-1}\vec{b} & \mathbf{A}^{N-2}\vec{b} & \cdots & \mathbf{A}\vec{b} & \vec{b} \\ | & | & \cdots & | & | \end{bmatrix}$ has to be all of $\mathbb{R}^4$.

9. Page Rank

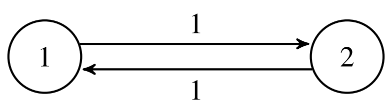
Learning Goal: *This problem highlights the use of transition matrices in modeling dynamical linear systems. Predictions about the steady state of a system can be made using the eigenvalues and eigenvectors of this matrix.*

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the setting of web traffic where the dynamical system can be described with a directed graph, also known as state transition diagram.

As we have seen in lecture and discussion the "transition matrix", $\mathbf{T}$, can be constructed using the state transition diagram, as follows: entries t_{ji}, represent the *proportion* of the people who are at website i that click the link for website j.

The **steady-state frequency** (i.e. fraction of visitors in steady-state) for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph. Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When the elements of this eigenvector are made to **sum to one** (to conserve population), the i^{th} element of the eigenvector will correspond to the fraction of people on the i^{th} website.

(a) For graph A shown below, what are the steady-state frequencies i.e. fraction of visitors in steady-state for the two webpages? Graph A has weights in place to help you construct the transition matrix. Remember to ensure that your steady state-frequencies sum to 1 to maintain conservation.



Graph A

Solution:

To determine the steady-state frequencies for the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state.

The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

To determine the eigenvalues of this matrix:

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0 \tag{9}$$

$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

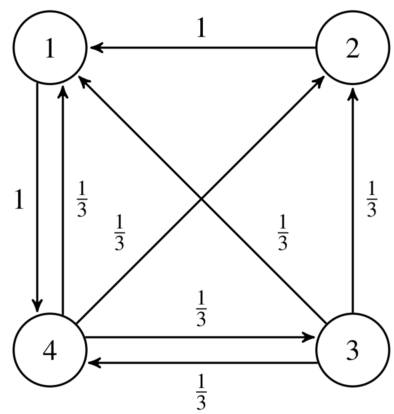
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{10}$$

The sum of the values of the vector should equal 1 since the number of people is conserved, so our conditions are:

$$v_1 + v_2 = 1$$
$$v_1 = v_2$$

The steady-state frequency eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has a steady-state frequency of 0.5.

(b) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this. It may be helpful to consult the Python documentation for `numpy.linalg.eig` to understand what this function does and what it returns. Graph B is shown below, with weights in place to help you construct the transition matrix.



Graph B

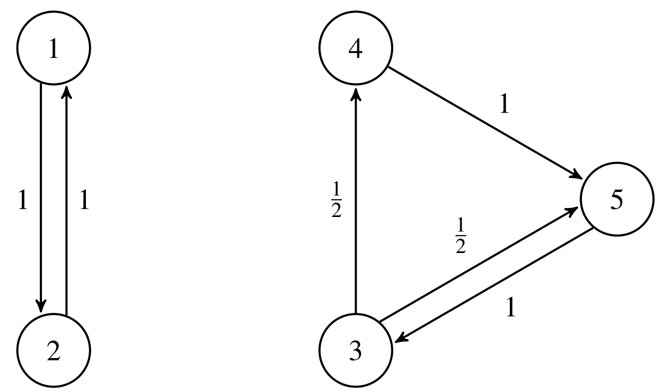
Solution:

To determine the steady-state frequencies, we need to create the transition matrix $\mathbf{T}$ first.

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

The eigenvector associated with eigenvalue 1 is $\begin{bmatrix} -0.61 & -0.31 & -0.23 & -0.69 \end{bmatrix}^T$ (found using IPython).
Scaling it appropriately so the elements add to 1, we get $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}^T$
These are the steady-state frequencies for the pages.

(c) Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C? Again, graph C with weights in place is shown below. You may use IPython to compute the eigenvalues and eigenvectors again.



Graph C

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

The eigenvalues of this graph are $\lambda = 1, 1, -1, -\frac{1}{2} - \frac{i}{2}, -\frac{1}{2} + \frac{i}{2}$ (found using IPython). The eigenspace associated with $\lambda = 1$ is span by the vectors $\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}^T$ and $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{bmatrix}^T$. While any linear combination of these vectors is an eigenvector, these two particular vectors have a nice interpretation.

The first eigenvector describes the steady-state frequencies for the last three webpages, and the second vector describes the steady-state frequencies for the first two webpages. This makes sense since there are essentially "two internets", or two disjoint sets of webpages. Surfers cannot transition between the two, so you cannot assign steady-state frequencies to webpage 1 and webpage 2 relative to the rest. This is why the eigenspace corresponding to the steady-state has dimension 2.

Assuming that each set of steady-state frequencies needs to add to 1, the first assigns steady-state frequencies of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns steady-state frequencies of 0.5 to both webpage 1 and webpage 2.

10. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.