EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 1

1. **Reading Assignment**

   For this homework, please read Note 0, Note 1A, and Note 1B. This will provide an overview of linear equations and augmented matrices. You are always welcome and encouraged to read ahead beyond this as well. Write a few sentences about how this relates to what you have learned before and what is new.

2. **Reading Reflection**

   Our modern world is filled with information devices and systems, and we want to get you thinking about them! Think about your favorite devices. If you're stuck, here are a few examples: cell phone camera, voice-activated speaker, heart rate monitor, vocal microphone, RADAR scanner. Write a short paragraph identifying the following:

   (a) How does this device physically sense and measure the real world?
   (b) What does it do with the information it senses?
   (c) Does this device have any actuators or other ways to respond what it senses?
   (d) What are some applications where this device is used? Are there any alternate options we can use?
   (e) What do you hope to learn about this device in this class?

   We hope this inspires your learnings and curiosities in this class!

   Solution: This question is graded purely on effort. If you have a particular curiousity, feel free to ask a TA! Many of us are doing intense research on these devices.

3. **Counting Solutions**

   Learning Goal: *(This problem is designed to illustrate the different types of systems of equations. Some sub-parts will have a unique solution and others have no solutions or infinitely many solutions. In this class, we will build up the mathematical machinery to systematically determine which case applies.)*

   Directions: For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. If there is a unique solution, find it. If there is an infinite number of solutions, explicitly state this, describe the set of all solutions, and then write one of such solutions. If there is no solution, explain why. **Show your work**.

   Example: We first provide an example to show two ways to solve systems of linear equations. In the first method, we will manipulate the equations directly. The second approach uses Gaussian elimination. Note that these methods are equivalent. **You may use either approach to solve the following problems. You will have more practice problems on Gaussian elimination in this homework, discussion, and future homeworks.**

$$\begin{aligned} 2x &+ 3y &= 5 \\ x &+ y &= 2 \end{aligned}$$

Solution A

$$2x + 3y = 5 \quad (1)$$
$$x + y = 2 \quad (2)$$

Subtract: (1) - 2*(2)

$$y = 1 \quad (3)$$

Now we plug in (3) into (2) and solve for x

$$x + 1 = 2$$
$$\rightarrow \quad x = 1 \quad (4)$$

From (3) and (4), we get the unique solution:

$$x = 1$$
$$y = 1$$

Solution B

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 1 & 1 & 2 \end{array}\right] \text{using } R_1 \leftarrow \tfrac{1}{2}R_1$$
$$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1$$
$$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 \end{array}\right] \text{using } R_2 \leftarrow -2R_2$$
$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \text{using } R_1 \leftarrow R_1 - \tfrac{3}{2}R_2$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a)

$$\begin{array}{ccccccc} x & + & y & + & z & = 3 \\ 2x & + & 2y & + & 2z & = 5 \end{array}$$

Solution:

Solution A:

$$x + y + z = 3 \quad (5)$$
$$2x + 2y + 2z = 5 \quad (6)$$

Subtract: (6) - 2*(5)

$$0 = -1 \quad (7)$$

We see this results in a contradiction in (7), indicating that no values of x,y,z can satisfy both equations. Therefore there are no solutions.

Solution B:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$

No solution. The fact that there are fewer equations than there are unknowns immediately means that it is not possible to have a unique solution; however, this does not guarantee that there is a solution to begin with. From Gaussian Elimination, we can see that these equations are contradictory since $0 \neq -1$. In other words, no values of x, y, and z can satisfy both equations simultaneously.

(b)

$$\begin{array}{rcrcrl} & - & y & + & 2z & = 1 \\ 2x & & & + & z & = 2 \end{array}$$

Solution:

Solution A:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

We notice that because we cannot cancel out x or y using the other equation, the equations do not contradict each other so there must exist an infinite number of solutions. We choose z to be our free variable and can then solve each equation in terms of z.

$$x = 1 - \frac{1}{2}z$$
$$y = 2z - 1$$

Solution B:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \end{array}\right] \text{swapping } R_1 \text{ and } R_2$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & -1 & 2 & 1 \end{array}\right] \text{using } R_1 \leftarrow \tfrac{1}{2}R_1$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & -1 \end{array}\right] \text{using } R_2 \leftarrow -R_2$$

We have now completed Gaussian elimination because we have a leading 1 in each row with zeros below that 1 in its column. In this way we can explicitly see that z is a free variable (x and y depend on z and there are no constraints on the value of z). Thus there are an infinite number of solutions. The set of infinite solutions has the form (for some $z \in \mathbb{R}$):

$$x = 1 - \frac{1}{2}z$$
$$y = 2z - 1$$

To get full credit it is enough to state "Infinite solutions" and give one possible solution that fits the form above.

(c)

$$\begin{aligned} x &+ 2y &= 5 \\ 2x &- y &= 0 \\ 3x &+ y &= 5 \end{aligned}$$

Solution:

Solution A:

In this case, there are three equations with only two unknowns. However, this fact alone does not tell us whether there is a unique solution, no solution, or an infinite number of solutions.

$$x + 2y = 5 \tag{8}$$
$$2x - y = 0 \tag{9}$$
$$3x + y = 5 \tag{10}$$

Adding (8) and (9), we obtain

$$3x + y = 5 \tag{11}$$

Notice that equation (11) = equation (10)! Put another way, equation (10) provides no new information about the system that equations (8) and (9) could not tell us (importantly, it also does not contradict any information from the previous equations as well). Knowing this, we focus only on (8) and (9).

Add (8) + 2*(9)

$$5x = 5$$
$$\rightarrow \quad x = 1 \tag{12}$$

Plugging this value of x back into (8), we obtain

$$1 + 2y = 5$$
$$\rightarrow \quad y = 2 \tag{13}$$

Yielding the unique solution

$$x = 1$$
$$y = 2$$

Solution B:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 3 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 3 & 1 & 5 \end{array}\right] \text{ using } R_2 \leftarrow R_2 - 2R_1$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{array}\right] \text{ using } R_3 \leftarrow R_3 - 3R_1$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & -10 \end{array}\right] \text{ using } R_2 \leftarrow -\tfrac{1}{5}R_2$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \text{ using } R_3 \leftarrow R_3 + 5R_2$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \text{ using } R_1 \leftarrow R_1 - 2R_2$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The system of linear equations at the end of the Gaussian Elimination above simply reads out

$$x = 1$$
$$y = 2$$
$$0 = 0$$

(d)

$$\begin{array}{rcrl} x & + & 2y & = 3 \\ 2x & - & y & = 1 \\ x & - & 3y & = -5 \end{array}$$

Solution:

Solution A:

$$x + 2y = 3 \tag{14}$$
$$2x - y = 1 \tag{15}$$
$$x - 3y = -5 \tag{16}$$

Add: (14) + (16)

$$2x - y = -2 \tag{17}$$

Subtract: (15) - (17)

$$0 = 3$$

This is a contradiction, so there is no solution.

There is no solution for this system as no choice of x and y can satisfy all equations simultaneously. This is often what happens when you have more equations than unknowns, although as you saw in the previous part, it doesn't always happen.

Solution B:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 1 & -3 & -5 \end{array}\right] \text{using } R_2 \leftarrow R_2 - 2R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -8 \end{array}\right] \text{using } R_3 \leftarrow R_3 - R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -8 \end{array}\right] \text{using } R_2 \leftarrow -\tfrac{1}{5}R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right] \text{using } R_3 \leftarrow R_3 + 5R_2$$

No solution. We can think of this to mean that there are no values of x and y which satisfy the conditions in all three equations simultaneously, because in order to satisfy all three equations, the last row $0 = -3$ would need to be true.

4. Matrix-Vector Multiplication

Learning Objective: *Practice evaluating matrix-vector multiplication.*

Complete the matrix-vector multiplication. If the product is not defined and thus has no solution, state this and justify your reasoning.

(a)

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1*1+0*2 \\ 2*1+1*2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution: No solution. The number of columns in the matrix does not match the number of elements (aka the number of rows) in the column vector.

(c)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 * \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 * \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 * \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

5. Word Problems

Learning Objective: *Understand how to setup a system of equations from word problems and solve using Gaussian elimination.*

For these word problems, represent the system of equations as an augmented matrix and solve. **We explicitly want you to use Gaussian elimination to solve.**

(a) Gustav is collecting soil samples. Each soil sample contains some sand, some clay, and some organic material. He wants to know the density of each material. His first sample has 0.5 liters of sand, 0.25 liters of clay, and 0.25 liters of organic material, and weighs 1.625 kg. His second sample contains 1 liter of sand, 0 liters of clay, and 1 liter of organic material, and weighs 3 kg. His third sample contains 0.25 liters of sand, 0.5 liters of clay, and 0 liters of organic material, and weighs 1.375 kg. That is,

$$0.5s + 0.25c + 0.25m = 1.625 \tag{18}$$

$$1s + 0c + 1m = 3 \tag{19}$$

$$0.25s + 0.5c + 0m = 1.375 \tag{20}$$

where s is the density of sand, c is the density of clay, and m is the density of organic material, all measured in kg/L. Solve for the density of each material.

Solution: We translate the system of equations to an augmented matrix and solve.

$$\left[\begin{array}{ccc|c} 0.5 & 0.25 & 0.25 & 1.625 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \text{using } R_1 \leftarrow 2R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 1 & 2 & 0 & 5.5 \end{array}\right] \text{using } R_3 \leftarrow 4R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0 & 1.5 & -0.5 & 2.25 \end{array}\right] \text{using } R_3 \leftarrow R_3 - R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_3 \leftarrow R_3 + 3R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0 & -1 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_2 \leftarrow R_2 - \tfrac{1}{2}R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_2 \leftarrow -2R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_1 \leftarrow R_1 - \tfrac{1}{2}R_2 - \tfrac{1}{2}R_3$$

We find sand has a density of 1.5 kg/L, clay has a density of 2 kg/L, and organic material has a density of 1.5 kg/L.

(b) Alice buys 3 apples and 4 oranges for 17 dollars. Bob buys 1 apple and 10 oranges for 23 dollars (Bob really likes oranges). How much do apples and oranges cost individually?

Solution: Treat x as the cost of one apple and y as the cost of one orange.

$$\begin{bmatrix} 3 & 4 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 23 \end{bmatrix}$$

Solving with Gaussian elimination, we get:

$$\left[\begin{array}{cc|c} 3 & 4 & 17 \\ 1 & 10 & 23 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 1 & 10 & 23 \end{array}\right] \text{using } R_1 \leftarrow \tfrac{1}{3}R_1$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 0 & \frac{26}{3} & \frac{52}{3} \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_1$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 0 & 1 & 2 \end{array}\right] \text{using } R_2 \leftarrow \tfrac{3}{26}R_2$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \text{using } R_1 \leftarrow R_1 - \tfrac{4}{3}R_2$$

We find apples cost \$3 each and oranges cost \$2 each.

(c) Jack, Jill, and James are driving from Berkeley to Las Vegas. Each of them takes a different route. Jack takes a short route and ends up going through Toll Road A and Toll Road B, costing him \$10. Jill takes a slightly longer route and goes through Toll Road B and Toll Road C, costing her \$15. Finally, James takes a wrong turn and takes Toll Road A twice, then takes Toll Road B and finally Toll Road C, costing him \$25. What is the toll cost on each road?

Solution: Treat a, b, c as the toll cost on each of Toll Roads A, B, and C respectively.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 25 \end{bmatrix}$$

Solving with Gaussian elimination, we get:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 2 & 1 & 1 & 25 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & -1 & 1 & 5 \end{array}\right] \text{using } R_3 \leftarrow R_3 - 2R_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 2 & 20 \end{array}\right] \text{using } R_3 \leftarrow R_3 + R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_3 \leftarrow \tfrac{1}{2}R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_2 \leftarrow R_2 - R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_1 \leftarrow R_1 - R_2$$

We find Toll Road A cost \$5, Toll Road B cost \$5, and Toll Road C cost \$10.

6. Ball weights

Learning Objective: *Understand how to setup a system of equations from word problems.*

For these word problems, you only need to setup the problem with Gaussian elimination or matrix-vector notation. Of course, you may solve for practice, but no additional credit is awarded.

Solution: Full credit is awarded for setting up the augmented matrix or matrix-vector notation correctly.

Your company wants to build a machine to automatically weigh and sort bowling balls. Your boss has asked you to build a demonstration using just 4 balls. She has given you a scale and 4 colored balls: blue, green, red, and black. These balls have weights w_b, w_g, w_r, and w_k respectively. Your goal is to determine the weight of each ball.

The simplest way you can determine the ball weights is to weigh them one at at time. That is, you will first weigh only the blue ball, then only the green ball, etc, in the order given above. After 4 measurements, the scale outputs 4 values corresponding to each measurement: y_1, y_2, y_3, and y_4. The augmented matrix setup would look like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_1 \\ 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 1 & 0 & y_3 \\ 0 & 0 & 0 & 1 & y_4 \end{array}\right]$$

Alternatively, the matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_b \\ w_g \\ w_r \\ w_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

(a) As you manually put the balls on the scale, you make a mistake and accidentally measure the balls in this order: red, black, blue, green. The scale outputs 4 values: a_1, a_2, a_3, a_4. Keeping the vector order the same as above (w_b, w_g, w_r, w_k), what would the augmented matrix look like?

Solution: One possible augmented matrix solution looks like this:

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 & a_2 \\ 1 & 0 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 & a_4 \end{array}\right]$$

Alternatively, the matrix-vector setup looks like this:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_b \\ w_g \\ w_r \\ w_k \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Note that as long as the ball vector order stays the same, we can technically rearrange the masks and measured value order as we like. For example, the following solutions are also correct:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 & a_4 \\ 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 & a_2 \end{array}\right] \quad \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & a_2 \\ 0 & 0 & 1 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_4 \\ 1 & 0 & 0 & 0 & a_3 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 1 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 1 & a_2 \end{array}\right]$$

Full credit is given for any correct answer.

The takeaway here is recognizing how the different ordering of masks may affect the ordering of your measurements, as in real life, you often can only change the order of the measurements and not your variables. Recognizing these misorderings can help you debug similar problems in the future, e.g. in lab.

(b) Your boss has now given you a robotic arm to pick up balls and put them on the scale. While setting up the code, you forget to tell the robot to remove the previous ball(s) already on the scale from the previous measurement. As a result, measurement one contains the blue ball, measurement two contains the blue ball and the green ball, etc. The outputs of your 4 measurements are z_1, z_2, z_3, and z_4 respectively. Setup the matrix problem for this.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & z_1 \\ 1 & 1 & 0 & 0 & z_2 \\ 1 & 1 & 1 & 0 & z_3 \\ 1 & 1 & 1 & 1 & z_4 \end{array}\right]$$

Alternatively, the matrix-vector setup looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_b \\ w_g \\ w_r \\ w_k \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

(c) While working from home, your cat accidentally jumps on your keyboard, causing a new code mistake. Now, during each measurement, instead of the robot putting one ball on the scale, it places the other 3 balls on the scale. For example, instead of measuring just the blue ball, the robot measures the green, red, and black balls together. The output of the 4 measurements are w_1, w_2, w_3, and w_4. Setup the matrix problem for this.

Solution: The augmented matrix setup looks like this:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & w_1 \\ 1 & 0 & 1 & 1 & w_2 \\ 1 & 1 & 0 & 1 & w_3 \\ 1 & 1 & 1 & 0 & w_4 \end{array}\right]$$

Alternatively, the matrix-vector setup looks like this:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_b \\ w_g \\ w_r \\ w_k \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.

