EECS 16A Designing Information Devices and Systems I
Spring 2021 Homework 2

This homework is due February 5, 2021, at 23:59.
Self-grades are due February 8, 2021, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any hand-written answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. **Reading Assignment**

   For this homework, please review Note 1B and read Note 2A. They will provide an overview of Gaussian elimination, vectors, and matrices. You are always welcome and encouraged to read beyond this as well, in particular, a quick look at Note 3 will help you. Describe how Gaussian elimination can help you understand if there are no solutions to a particular system of equations? What about a unique solution? Does a row of zeros always mean there are infinite solutions?

2. **Gaussian Elimination**

   Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

   (a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

      i. Draw the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \tag{1}$$
$$2x - 4y = 4 \tag{2}$$
$$3x - 2y = 8 \tag{3}$$

      ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

      iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination.

$$\begin{aligned} x+2y+5z &= 3 \\ x+12y+6z &= 1 \\ 2y+z &= 4 \\ 3x+16y+16z &= 7 \end{aligned}$$

(c) Consider the following system of equations:

$$\begin{aligned} x+2y+5z &= 6 \\ 3x+9y+6z &= 3 \end{aligned}$$

You are given a set S of candidate solutions,

$$S = \left\{ \vec{v} \;\middle|\; \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} -11 \\ 3 \\ 1 \end{bmatrix} t \quad , \quad t \in \mathbb{R} \right\}$$

This vector notation can be expressed in terms of its components:

$$\vec{v} = \begin{bmatrix} 16-11t \\ -5+3t \\ t \end{bmatrix} \quad \text{means} \quad \begin{aligned} x &= 16-11t \\ y &= -5+3t \\ z &= t \end{aligned}$$

Show, by substitution, that any $\vec{v} \in S$ is a solution to the system of equations given above. Note that this means that the candidate solution must satisfy the system of equations for all $t \in \mathbb{R}$.

(d) Consider the following system:

$$\begin{aligned} 4x+4y+4z+w+v &= 1 \\ x+y+2z+4w+v &= 2 \\ 5x+5y+5z+w+v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Determine the solutions to the set of equations.

3. Linear Dependence

Learning Goal: *Evaluate the linear dependency of a set of vectors.*

State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

(a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

4. Filtering Out The Troll

Learning Goal: *The goal of this problem is to represent a practical scenario using a simple model of directional microphones. You will tackle the problem of sound reconstruction through solving a system of linear equations.*

You attended a very important public speech and recorded it using a recording device that had two directional microphones. However, there was a person in the audience who was trolling around, adding interference to the recording. When you went back home to listen to the recording, you realized that the two recordings were dominated by the troll's interference and you could not hear the speech. Fortunately, since your recording device contained two microphones, you realized there is a way to combine the two individual microphone recordings so that the troll's interference is removed. You remembered the locations of the speaker and the troll and created the diagram shown in Figure 1. You (and your two microphones) are located at the origin.

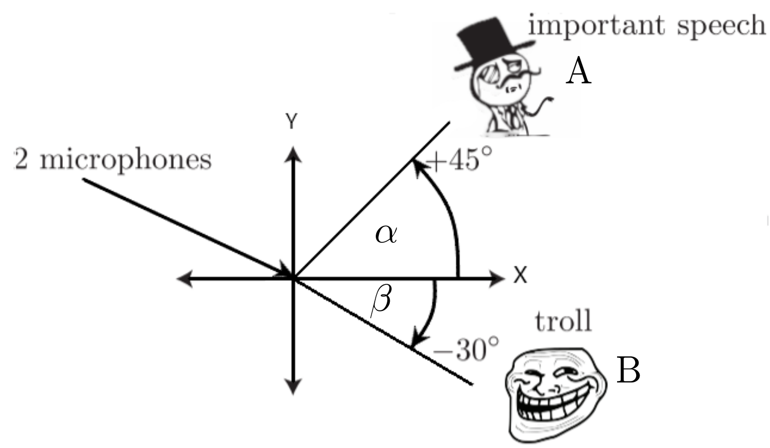

Figure 1: Locations of the speaker and the troll.

Each directional microphone records signals differently based on their angles of arrival. The first microphone weights (multiplies) a signal, coming from an angle θ with respect to the x-axis, by the factor $f_1(\theta) = \cos(\theta)$. If two signals are simultaneously playing (as is the case with the speech and the troll interference), then a linear combination (i.e. a weighted sum) of both the signals is recorded, each weighted by the respective $f_1(\theta)$ for their angles. The second microphone weights a signal, coming from an angle θ with respect to the x-axis, by the factor $f_2(\theta) = \sin(\theta)$. Again, if there are two simultaneous signals, then the second microphone also records the weighted sum of both the signals.

For example, an audio source that lies on the x axis will be recorded by the first microphone with weight equal to 1 (since $\cos(0) = 1$), but will not be recorded up by the second microphone (since $\sin(0) = 0$). Note that the weights can also be negative.

Let us represent the speech sample at a particular time-instant by the variable a and the interference caused by the troll at the same time-instant by the variable b. Remember, we do not know either a or b. The recording of the first microphone at that time instant is given by m_1:

$$m_1 = f_1(\alpha) \cdot a + f_1(\beta) \cdot b,$$

and the second microphone recorded the signal

$$m_2 = f_2(\alpha) \cdot a + f_2(\beta) \cdot b.$$

where α and β are the angles at which the public speaker A and the troll B respectively are located with respect to the x-axis, and variables a and b are the audio signals produced by the public speaker A and the troll B respectively.

(As a side note, we could represent the entire speech with a vector $\vec{a}$ by stacking all the speech samples on top of each other, and this is what we would typically do in a real-world speech processing example. However, here we consider just one time instant of the speech for simplicity.)

(a) Plug in the values of α and β to write the recordings of the two microphones m_1 and m_2 as a linear combination (i.e. a weighted sum) of a and b.

(b) Solve the system you wrote out on the earlier part to recover the important speech a, as a weighted combination of m_1 and m_2. In other words, write $a = u \cdot m_1 + v \cdot m_2$ (where u and v are scalars). What are the values of u and v?

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

5. [PRACTICE/OPTIONAL] Tyler's Optimal Tea

Learning Goal: *Recognize a problem that can be cast as a system of linear equations.*

Tyler's Optimal Tea has a unique way of serving its customers. To ensure the best customer experience, each customer gets a combination drink personalized to their tastes. Tyler knows that a lot of customers don't know what they want, so when customers walk up to the counter, they are asked to taste four standard combination drinks that each contain a different mixture of the available pure teas.

Each combination drink (Classic, Roasted, Mountain, and Okinawa) is made of a mixture of pure teas (Black, Oolong, Green, and Earl Grey), with the total amount of pure tea in each combination drink always the same, and equal to one cup. The table below shows the quantity of each pure tea (Black, Oolong, Green, and Earl Grey) contained in each of the four standard combination drinks (Classic, Roasted, Mountain, and Okinawa).

Tea [cups]	Classic	Roasted	Mountain	Okinawa
Black	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$
Oolong	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{3}$
Green	0	$\frac{1}{3}$	$\frac{3}{5}$	0
Earl Grey	$\frac{1}{3}$	0	0	0

Initially, the customer's ratings for each of the pure teas are unknown. Tyler's goal is to determine how much the customer likes each of the pure teas, so that an optimal combination drink can then be made. By letting the customer taste and score each of the four standard combination drinks, Tyler can use linear algebra to determine the customer's initially unknown ratings for each of the pure teas. After a customer gives a score (all of the scores are real numbers) for each of the four standard combination drinks, Tyler then calculates how much the customer likes each pure tea and mixes up a special combination drink that will maximize the customer's score.

The score that a customer gives for a combination drink is a linear combination of the ratings of the constituent pure teas, based on their proportion. For example, if a customer's rating for black tea is 6 and oolong tea is 3, then the total score for the Okinawa Tea drink would be $6 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 5$ because Okinawa has $\frac{2}{3}$ black tea and $\frac{1}{3}$ Oolong tea.

Professor Waller was thirsty after giving the first lecture, so Professor Waller decided to take a drink break at Tyler's Optimal Tea. Professor Waller walked in and gave the following ratings:

Combination Drink	**Score**
Classic	7
Roasted	7
Mountain	$7\frac{2}{5}$
Okinawa	$6\frac{1}{3}$

(a) What were Professor Waller's ratings for each tea? **Work this problem out by hand in terms of the steps. You may use a calculator to do algebra.**

(b) What mystery tea combination could Tyler put in Professor Waller's personalized drink to maximize the customer's score? If there is more than one correct answer, state that there are many answers, and give one such combination. What score would Professor Waller give for the answer you wrote down? Assume the total amount of tea must be one cup.

6. Fountain Codes

Learning Goal: *Linear algebra shows up in many important engineering applications. Wireless communication and information theory heavily rely on principles of linear algebra. This problem illustrates some of the techniques used in wireless communication.*

Alice wants to send a message to her friend Bob. Alice sends her message $\vec{m}$ across a wireless channel in the form of a transmission vector $\vec{w}$. Bob receives a vector of symbols denoted as $\vec{r}$.

Alice knows some of the symbols in the transmission vector that she sends may be corrupted, so she needs a way to protect her message from the corruptions. (Transmission corruptions occur commonly in real wireless communication systems, for instance, when your laptop connects to a WiFi router, or your cellphone connects to the nearest cell tower.) Ideally, Alice can come up with a transmission vector such that if some of the symbols get corrupted, Bob can still figure out what Alice is trying to say!

One way to accomplish this goal is to use fountain codes, which are part of a broader family of codes called error correcting codes. The basic principle is that instead of sending the exact message, Alice sends a modified longer version of the message so that even if some parts are corrupted Bob can recover what she meant. Fountain codes are based on principles of linear algebra, and were actually developed right here at Berkeley! The company that commercialized them, Digital Fountain, (started by a **Berkeley grad, Mike Luby**), was later acquired by Qualcomm. In this problem, we will explore some of the underlying principles that make fountain codes work in a very simplified setting.

The message that Alice wants to send to Bob is the three numbers a, b, and c. The message vector representing these numbers is $\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Figure 2 shows how Alice's message is encoded in a transmission vector and how Bob's received vector may have some corrupted symbols.

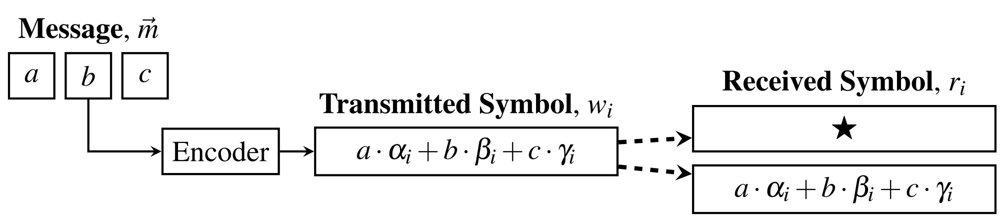

Figure 2: Each symbol in a transmission vector $\vec{w}$ is a linear combination of a, b, and c. Each transmitted symbol, w_i, is either received exactly as it was sent, or it is corrupted. A corrupted symbol is denoted by ★. The ith row of a symbol generating matrix G determines the values of α_i, β_i, and γ_i.

(a) Since Alice has three numbers she wishes to send, she could transmit six symbols in her transmission vector for redundancy. This transmission strategy is called the "repetition code".

If Alice uses the repetition code, her transmission vector is $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$.

As depicted in Figure 2, the received vector may have corrupted symbols. For example, suppose only the first symbol was corrupted, then Bob would receive the vector $\vec{r} = \begin{bmatrix} \bigstar \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$, where the $\bigstar$ symbol represents a corrupted symbol.

Using the repetition code scheme, give an example of a received vector $\vec{r}$ with only two corrupted symbols such that a is unrecoverable but b and c are still recoverable.

(b) Alice can generate $\vec{w}$ by multiplying her message $\vec{m}$ by a matrix. Write a matrix-vector multiplication that Alice can use to generate $\vec{w}$ according to the repetition code scheme. Specifically, find a "generating" matrix G_R such that $G_R\vec{m} = \vec{w}$, where $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$.

(c) Instead of a repetition code, it is also possible to use other codes (e.g. fountain codes). Alice and Bob can choose any symbol generating matrix, as long as they agree upon it in advance. Each different matrix represents a different "code." Alice's TA recommends using the symbol generating matrix G_F:

$$G_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Alice then uses the symbol generating matrix G_F to produce a new transmission vector: $G_F\vec{m} = \vec{w}$.

Suppose Bob receives the vector $\vec{r} = \begin{bmatrix} 7 \\ \bigstar \\ \bigstar \\ 3 \\ 4 \\ \bigstar \\ \bigstar \end{bmatrix}$, which is a corrupted version of $\vec{w}$.

Write a system of linear equations that Bob can use to recover the message vector $\vec{m}$. Solve it to recover the three numbers that Alice sent.

Hint: Consider the rows of G_F that correspond to the uncorrupted symbols in $\vec{r}$.

(d) We explore one example case where Alice and Bob agree to use G_F (i.e. $G_F\vec{m} = \vec{w}$) and there are three corruptions, so Bob receives four uncorrupted symbols.

Suppose Bob receives $\vec{r} = \begin{bmatrix} 1 \\ \bigstar \\ 3 \\ \bigstar \\ 4 \\ \bigstar \\ 9 \end{bmatrix}$. Can you determine the message $\vec{m}$ that Alice sent?

Note: it can be shown that receiving *any* four uncorrupted symbols when Alice is using G_F is enough to recover Alice's message. On the other hand, we showed in part (a) that receiving *any* four uncorrupted symbols using G_R does not guarantee we can recover Alice's message. This is why, in practice, we would prefer to use the fountain code G_F instead of the repetition code G_R — it is a more reliable way to send messages.

7. Show It!

Learning Goal: *This is an opportunity to practice your proof development skills. For proofs you have seen before, use this as an opportunity to make sure you understood them by trying them independently using the steps discussed in class.*

(a) **Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $\mathbf{A}$ are linearly dependent.**

This problem has 4 sub-parts and the following is a chart showing the sequential steps we are going to take to approach this proof.

In a textbook you might see the steps in a proof written out in the order in the middle column of the table. But when you are building a proof you usually want to go in another order — this is the order of the subparts in this problem.

Proof steps		**Corresponding problem sub-parts**
1	Write what is known	Sub-part (i)
2	Manipulate what is known	Sub-part (iii)
3	Connecting it up	Sub-part (iv)
4	What is to be shown	Sub-part (ii)

(i) **First Step: write what you know**

Think about the *information we already know* from the problem statement. We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions. Infinitely many solutions are hard to work with, but perhaps we can simplify to something that we can work with. If the system has infinite number of solutions, it must have at least ___ distinct solutions (Fill in the blank).

So let us assume that $\vec{u}$ and $\vec{v}$ are two different vectors, both of which are solutions to $\mathbf{A}\vec{x} = \vec{b}$. Express the sentence above in a mathematical form (Just writing the equations will suffice; no need to do further mathematical manipulation).

(ii) **What we want to show:**

Now consider *what we need to show*. We have to show that the columns of $\mathbf{A}$ are linearly dependent. Let us assume that $\mathbf{A}$ has columns $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$. Using the definition of linear dependence from **Note 3 Subsection 3.1.1**, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.

(iii) **Manipulating what we know:**
Now let us try to start from the **First step: equations from (i)**, make mathematically logical steps and reach the **What we want to show: equations from (ii)**. Since your answer to (ii) is expressed in terms of the column vectors of **A**, let us try to express the mathematical equations from (i), in terms of the column vectors too. For example, we can write

$$\mathbf{A}\vec{x} = \vec{b}$$

$$\implies \begin{bmatrix} | & | & \dots & | \\ \vec{c_1} & \vec{c_2} & \dots & \vec{c_n} \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \vec{b}$$

$$\implies x_1\vec{c_1} + x_2\vec{c_2} + \ldots + x_n\vec{c_n} = \vec{b}$$

Notice that x_1,... x_n etc are scalars. Now use your answer to part (i) to repeat the above formulation for distinct solutions $\vec{u}$ and $\vec{v}$.

(iv) **Connecting it up:**
Now think about how you can mathematically manipulate your answer from part (iii) (**Manipulating what we know**) to **match the pattern** of your answer from part (ii) (**What we want to show**).

(b) Now try this proof on your own. Similar proofs will also be covered in your discussion section 2A. Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

(c) Let n be a positive integer. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of k linearly dependent vectors in $\mathbb{R}^n$. Show that for *any* $n \times n$ matrix $\mathbf{A}$, the set $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2, \ldots, \mathbf{A}\vec{v}_k\}$ is a set of linearly dependent vectors.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.