EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 3B

1. Mechanical Inverses

For each sub-part below, determine whether or not the inverse of **A** exists.
If it exists, compute the inverse using Gauss-Jordan method.

$A^{-1}A = I$

$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$

$\vec{x} = I\vec{x} = A^{-1}\vec{b}$

$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \Rightarrow \left[\begin{array}{c|c} I & \vec{x} \end{array}\right]$

$\left[\begin{array}{c|c} A & I \end{array}\right] \Rightarrow \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$

Gauss - Jordan

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$[A | I]$

$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/9 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right]$ $A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\downarrow$

$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/9 \end{array}\right] \frac{1}{9}R_2 = \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$

(b) (PRACTICE)
$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 4/5 & 1/5 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 4/5 & 1/5 & 0 \\ 0 & 1/5 & -1/5 & 1 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & 4/5 & 1/5 & 0 \\ 0 & 1 & -1 & 5 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 5 \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 1 & d/c & 0 & 1/c \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{d}{c} - \frac{b}{a} & -1/a & 1/c \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ general 2x2 inverse

(d) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & 4 \end{bmatrix}$ No inverse for non-square matrixes

2×3

(e) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$ $A\vec{x} = \vec{b}$ LD columns → multiple solns to $\vec{x}$, so we don't have a unique inverse/reverse mapping

The same ⇒ Linear dependence

No Inverse

(f) (PRACTICE)

$\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$ Not L.D. cols. Yes inv. exists $A^{-1} = \begin{bmatrix} -4/5 & 2 & 1 \\ 2/5 & -1/2 & -1 \\ 1/5 & -1/2 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1/2 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 0 & 0 & -1 & -1/5 & 1/2 & 0 \\ 0 & -1 & 1 & -1/5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 2/5 & -1/2 & -1 \\ 0 & -1 & 1 & -1/5 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 2/5 & -1/2 & -1 \\ 0 & 0 & 1 & 1/5 & -1/2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/5 & 2 & 1 \\ 0 & 1 & 0 & 2/5 & -1/2 & -1 \\ 0 & 0 & 1 & 1/5 & -1/2 & 0 \end{array}\right]$$

Vector Spaces in Matrixes

<u>subspaces</u>
(1) has zero vector
(2) closed under vector addition
(3) closed under scalar multiplication

$\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ ✓ satisfies all conditions

$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ just two vectors ✗ satisfies no conditions

<u>column space</u> of a matrix $col(A)$
$= \text{span}\{\text{column vectors of } A\}$

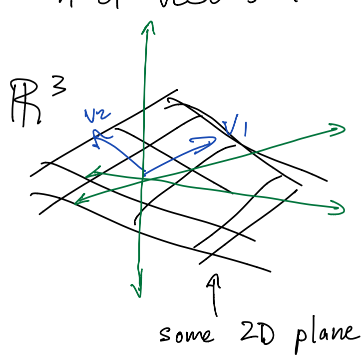
<u>rank</u> = # of LI column vectors
$= \dim(\text{span}(col(A)))$ ~ LI columns

<u>null space</u> $\{\vec{x} \mid A\vec{x} = 0\}$

$\dim(N(A)) = \#\text{ of cols in } A - \text{rank}(A)$
~ LD columns

<u>basis</u> The set $\{\vec{b}_1 \cdots \vec{b}_n\}$ is a basis for V if
(1) $\{\vec{b}_1 \cdots \vec{b}_n\}$ are LI
(2) for $\vec{v} \in V$, there exists some $c_1 \cdots c_n$ such that $\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$

# of vectors in basis = dimension



v_1, v_2 are the basis for a subspace (a plane) in $\mathbb{R}^3$

generally, bases are not unique.
(more in future lectures)

2. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix. span(col(A))
- The **null space** is the set of input vectors that output the zero vector. $A\vec{x}=0$

For the following matrices, answer the following questions:

i. What is the column space of **A**? What is its dimension?

ii. What is the null space of **A**? What is its dimension?

iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?

iv. Do the columns of **A** span $\mathbb{R}^2$? Do they form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

i) $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ 1 dimensional $\to$ 1 LI vector

ii) $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ soln: $x_1 = 0$, $x_2 = \alpha$ Null space $= \text{span}\left\{\begin{bmatrix}0\\\alpha\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ 1 dimensional

iii) Yes

iv) No, only 1 LI vector in column space

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

i) $\text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ 1 dimensional

ii) $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = \alpha$, $x_2 = 0$ Null space $= \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

iii) $\text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} \neq \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ No, not the same.

iv) No, only 1 LI vector. not span $\mathbb{R}^2$

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

i) $\text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\} = \mathbb{R}^2$ 2 LI vectors

ii) $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \vec{x} = \vec{0}$ Null space $= \text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ 0 dimensional

iii) Yes, still spans all of $\mathbb{R}^2$, even though we have different vectors / bases.

iv) Yes

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

i) $\text{span}\left\{\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$

ii) $\left[\begin{array}{cc|c} -2 & 4 & 0 \\ 3 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - 2x_2 = 0 \rightarrow x_1 = 2x_2$

Null space $= \text{span}\left\{\begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}\right\}$

$= \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

iii) No, different subspace $\sim \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

iv) No, only 1 LI vector

i) $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$

ii) $\left[\begin{array}{cccc|c} 1 & -1 & -2 & -4 & 0 \\ 1 & 1 & 3 & -3 & 0 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -4 & 0 \\ 0 & 2 & 5 & 1 & 0 \end{array}\right] -R_1$

$\downarrow$

$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -4 & 0 \\ 0 & 1 & 5/2 & 1/2 & 0 \end{array}\right] \tfrac{1}{2}R_2$

$\downarrow$

$\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & -7/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 0 \end{array}\right] +R_2 \rightarrow$

$x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0$

$x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0$

x_3, x_4 are free

Null space $= \begin{bmatrix} -\frac{1}{2}x_3 + \frac{7}{2}x_4 \\ -\frac{5}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$

$= \text{span}\left\{\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}\right\}$ 2D

iii) Still 2 LI vectors $\Rightarrow$ col(A) $= \mathbb{R}^2$ still spans the same subspace

iv) No, too many LD vectors $\Rightarrow$ not a basis.

3. Helpful Guide – Reference Definitions

Vector spaces:

A *vector space* V is a set of elements that is 'closed' under vector addition and scalar multiplication and contains a zero vector. What does closed mean?

That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

More formally, a *vector space* (V, F) is a set of vectors V, a set of scalars F, and two operators that satisfy the following properties:

As a reminder, the mathematical notation $\forall\ \vec{v},\vec{u},\vec{w} \in V$ means *for all possible vectors* $\vec{u},\vec{v},\vec{w}$ *within the vector space* V.

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w} \quad \forall \quad \vec{v},\vec{u},\vec{w} \in V$.
  - Commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u} \quad \forall \quad \vec{v},\vec{u} \in V$.
  - Additive Identity: There exists an additive identity $\vec{0} \in V$ such that $\vec{v}+\vec{0} = \vec{v} \quad \forall \quad \vec{v} \in V$.
  - Additive Inverse: For any $\vec{v} \in V$, there exists $-\vec{v} \in V$ such that $\vec{v}+(-\vec{v}) = \vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v} \quad \forall \quad \vec{v} \in V,\ \alpha,\beta \in F$.
  - Multiplicative Identity: There exists $1 \in F$ where $1 \cdot \vec{v} = \vec{v} \quad \forall \quad \vec{v} \in F$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v} \quad \forall \quad \alpha \in F$ and $\vec{u},\vec{v} \in V$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v} \quad \forall \quad \alpha,\beta \in F$ and $\vec{v} \in V$.

Subspaces:

A subset W of a *vector space* V is a *subspace* of V if the above conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace W.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

Basis:

A *basis* for a vector space or subspace is an *ordered set of linearly independent vectors* that *spans the vector space or subspace*.

Therefore, if we want to check whether a set of vectors $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ forms a basis for a vector space V, we check for two important properties:

(a) $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ is linearly independent.

(b) $\operatorname{span}\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\} = V$

As we move along, we'll learn how to identify and construct a basis, and we'll also learn some interesting properties of bases.

Dimension:

The *dimension* of a vector space is the *minimum number* of vectors needed to span the entire vector space. That is, the dimension of a vector space equals the number of vectors in a basis for this vector space.