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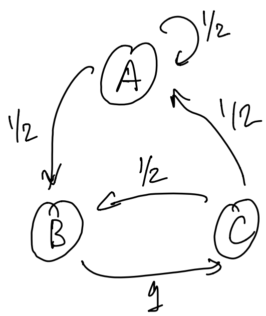
Transitions Matrices



$$x_A[t+1] = \tfrac{1}{2} x_A[t] + \tfrac{1}{2} x_C[t]$$
$$x_B[t+1] = \tfrac{1}{2} x_A[t] + \tfrac{1}{2} x_C[t]$$
$$x_C[t+1] = x_B[t]$$

$$\begin{bmatrix} x_A[t+1] \\ x_B[t+1] \\ x_C[t+1] \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A[t] \\ x_B[t] \\ x_C[t] \end{bmatrix}$$

$$\vec{x}[t+1] = P \quad \vec{x}[t]$$

$$\vec{x}[t+2] = P \cdot P \cdot \vec{x}[t] = P\,\vec{x}[t+1]$$

Invertibility

$P^{-1} \sim \quad P^{-1}P\vec{x}[t] = \vec{x}[t]$

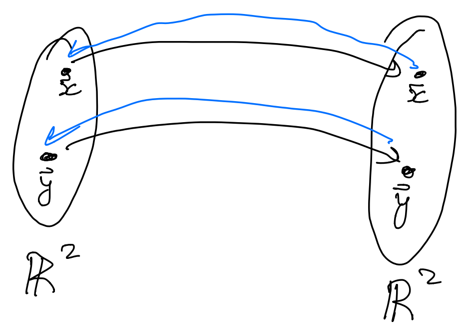
$P^{-1}\vec{x}[t+1] = \vec{x}[t]$

input $\leftarrow \boxed{A^{-1}} \leftarrow$ output

input $\rightarrow \boxed{A} \rightarrow$ output

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

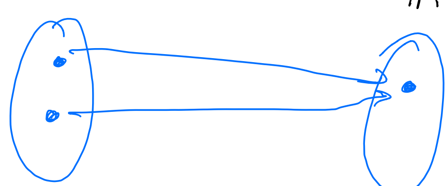
$$\underset{\substack{\uparrow \\ \text{input}}}{A\vec{x}} = \underset{\substack{\uparrow \\ \text{output}}}{\vec{x}}$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



2 inputs map to same output

No unique reversal / mapping

Calculating Inverse

$A\vec{x} = \vec{b}$ $\qquad \left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & \vec{x} \end{array}\right]$

$AM = I$ $\qquad \left[\begin{array}{c|ccc} A & 1 & 0 & \cdots \\ & 0 & 1 & \\ & 0 & 0 & \\ & \vdots & \vdots & \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$

$M = A^{-1}$

Theorem: If col(A) are LD $\Rightarrow$ A not invertible

If A is invertible $\rightarrow$ col(A) are LI

$A\vec{x} = \vec{0}$ $\qquad \underline{\vec{x} \neq 0}$

$\underset{\sim}{A^{-1}}A\vec{x} = A^{-1}\vec{0} = \vec{0}$

$I\vec{x} = \vec{0}$

$\underline{\vec{x} = 0}$

contradiction

If A not invertible $\Rightarrow$ col(A) are LD

If col(A) are LI $\Rightarrow$ A is invertible

EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 3A

1. Inverses

In general, the *inverse* of a matrix "undoes" the operation that a matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}\vec{b} = I\vec{x} = \vec{x}$

Properties of Inverses

For a matrix $\mathbf{A}$, if its inverse exists, then:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1} \quad \text{for a nonzero scalar } k \in \mathbb{R}$$

$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T} \quad T \text{ is "Transpose"}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad \text{assuming } \mathbf{A}, \mathbf{B} \text{ are both invertible}$$

$k^{-1} = 1/k$

(a) Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all invertible matrices. Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

$\cancel{C^{-1}B^{-1}A^{-1}ABC}\,(ABC)^{-1} = C^{-1}B^{-1}A^{-1}I$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(b) Now consider the following four matrices.

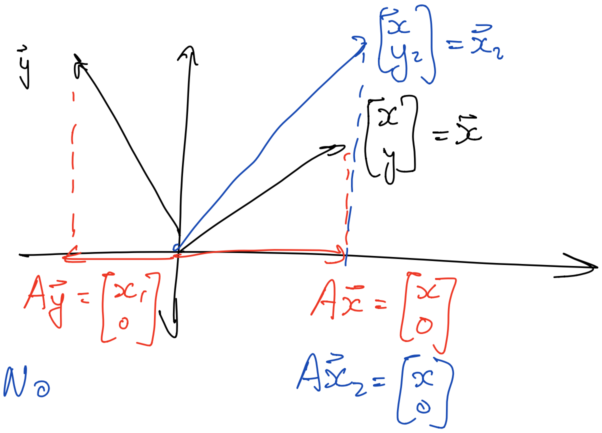
$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.
ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.
iii. Are the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ invertible?
iv. Can you find anything in common about the rows (and columns) of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$? (*Bonus:* How does this relate to the invertibility of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?)
v. Are all square matrices invertible?
vi. (PRACTICE) How can you find the inverse of a general $n \times n$ matrix?

see attached

1b) i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

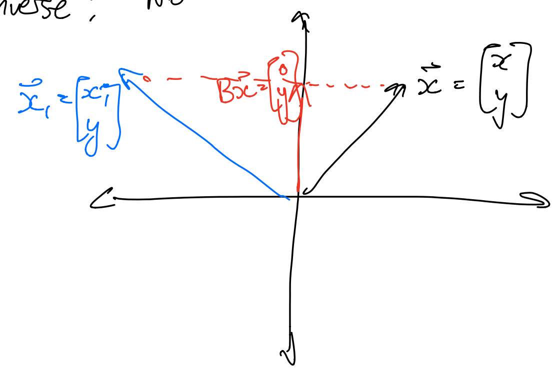
$A\vec{x} = \begin{bmatrix} x \\ 0 \end{bmatrix}$



ii) Can we undo? No

iii) Is there an inverse? No

$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

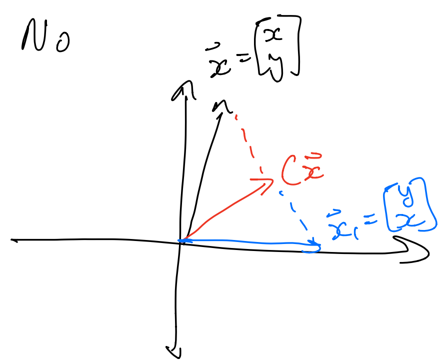


i) $B\vec{x} = \begin{bmatrix} 0 \\ y \end{bmatrix}$

ii) Can we undo? No

iii) Is there an inverse? No

$C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$



i) $C\vec{x} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \end{bmatrix}$

ii) Can we undo? No

iii) No inverse

iv) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\Rightarrow$ All have linearly dependent columns

$\Rightarrow$ All are not invertible

v) No, not all square matrices are invertible

2b) solve for S^{-1}

$$\left[\begin{array}{cc|cc} 3/4 & 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \times 4/3$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1 & 2 & 0 & 4 \end{array}\right] \times 4$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & 4/3 & -4/3 & 4 \end{array}\right] - R_1$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right] \div 4/3$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right] - \tfrac{2}{3}R_2$$

$$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

2e) 2 steps

method 1

$$x[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix} \quad S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

$$x[2] = Sx[1] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$$

$$x[3] = Sx[2] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix}$$

method 2

$$x[3] = (S \cdot S)x[1] = S^2 x[1]$$

$$S^2 = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}\begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix}$$

$$x[3] = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix}\begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix}$$

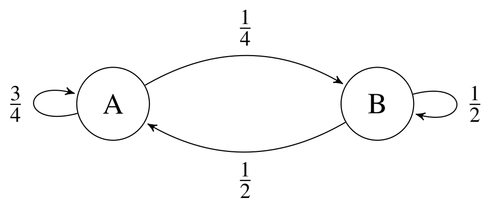
2. Transition Matrix

Suppose we have a network of pumps as shown in the diagram below. Let us describe the state of A and B using a state vector $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$ where $x_A[n]$ and $x_B[n]$ are the states at time-step n.

$$\vec{x}[1] = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

$\Sigma = 1 \quad \Sigma = 1$



$$\vec{x}[2] = \begin{bmatrix} \frac{3}{4}4 + \frac{1}{2}4 \\ \frac{1}{4}4 + \frac{1}{2}4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

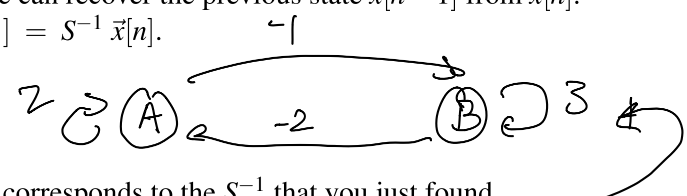
(a) Find the state transition matrix S, such that $\vec{x}[n+1] = S\,\vec{x}[n]$.
Separately find the sum of the terms for each column vector in S. Do you notice any pattern?

$\Rightarrow$ conservative when columns sum to 1

(b) Let us now find the matrix S^{-1} such that we can recover the previous state $\vec{x}[n-1]$ from $\vec{x}[n]$.
Specifically, solve for S^{-1} such that $\vec{x}[n-1] = S^{-1}\,\vec{x}[n]$.

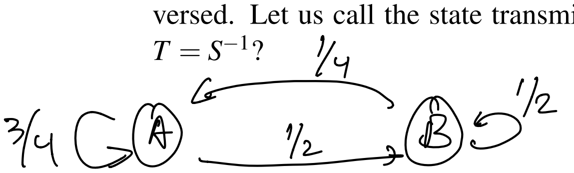
see attached for work

$$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$



(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found.
Also find the sum of the terms for each column vector in S^{-1}. Do you notice any pattern? still sum to 1, but no longer physically possible

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?



$$T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = S^T$$

reverse = transpose

(e) Suppose we start in the state $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$. Compute the state vector after 2 time-steps $\vec{x}[3]$.

See attached

(f) **(Challenge practice problem)** Given our starting state from the previous problem, what happens if we look at the state of the network after a lot of time steps? Specifically which state are we approaching, as defined below?

$$\vec{x}_{final} = \lim_{n\to\infty} \vec{x}[n]$$

Note that the final state needs to be what we call a *steady state*, meaning $\boxed{S\,\vec{x}_{final} = \vec{x}_{final}}$. steady state
Also what can you say about $x_A[n] + x_B[n]$?
Use information from both of these properties to write out a new system of equations and solve for $\vec{x}_{final}$.

2f) Steady state

$$S\vec{x} = \vec{x} = I\vec{x}$$

$$S\vec{x} - I\vec{x} = \vec{0}$$

$$(S-I)\vec{x} = \vec{0}$$

$$S-I = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} -1/4 & 1/2 \\ 1/4 & -1/2 \end{bmatrix}\vec{x} = \vec{0} \qquad \vec{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

$$x_A - 2x_B = 0$$

$$\vec{x} = \alpha\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \alpha\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}\left(\alpha\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}\right) = \begin{bmatrix} \alpha\, 2/4 + \alpha\, 1/6 \\ \alpha\, 1/6 + \alpha\, 1/6 \end{bmatrix} = \begin{bmatrix} \alpha\, 2/3 \\ \alpha\, 1/3 \end{bmatrix} = \alpha\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

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