Optional feedback form: tinyurl.com/eecs16a-sp21-6b

Linear Dependence / Independence

set of vectors $\{\vec{a}_1 \ldots \vec{a}_n\}$ is L.D. iff

Equivalent Defn.

$$\vec{a}_j = \sum_{m \neq j} \alpha_m \vec{a}_m \quad \text{with at least one } \alpha_m \neq 0$$

$$\vec{0} = \sum_{m} \alpha_m \vec{a}_m \quad \text{with at least one } \alpha_m \neq 0$$

Span of a set of vectors $\{\vec{a}_1, \ldots \vec{a}_n\}$

All vectors that satisfy

$$\vec{v} = \sum_i c_i \vec{a}_i \quad \underline{\text{linear combination of } \vec{a}_i}$$

$$= c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + \ldots + c_n\vec{a}_n$$

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{v} \\ | \end{bmatrix}$$

System of Linear Eqns. (see proof. lecture notes)

$$A\vec{x} = \vec{b}$$

① $A\vec{x} = \vec{b}$ has a soln iff $\vec{b} \in \text{span}(\text{col}(A))$

② $A\vec{x}$ is a linear comb. of $\text{col}(A)$

③ Unique soln $\triangleq$ unique $\vec{x}$ that satisfies $A\vec{x} = \vec{b}$

④ Every $\vec{b} \in \text{span}(\text{col}(A))$ is unique iff the columns of A are linearly independent.

if cols(A) L.D. $\Rightarrow$ $\underline{A\vec{x} = \vec{0}}$

Possible solns to system of Linear Eqns

case 1 : $\vec{b} \notin \text{span}(\text{col}(A))$ ~ no soln

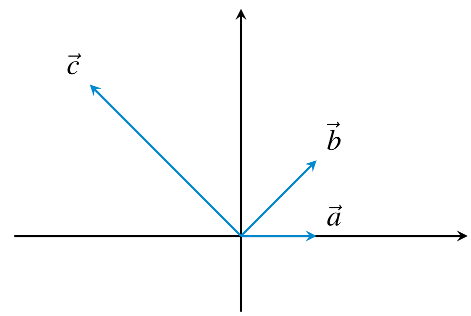
case 2 : $\vec{b} \in \text{span}(\text{col}(A))$ $\rightarrow$ L.I. cols. ~ unique

$\rightarrow$ L.D. cols. ~ inf.

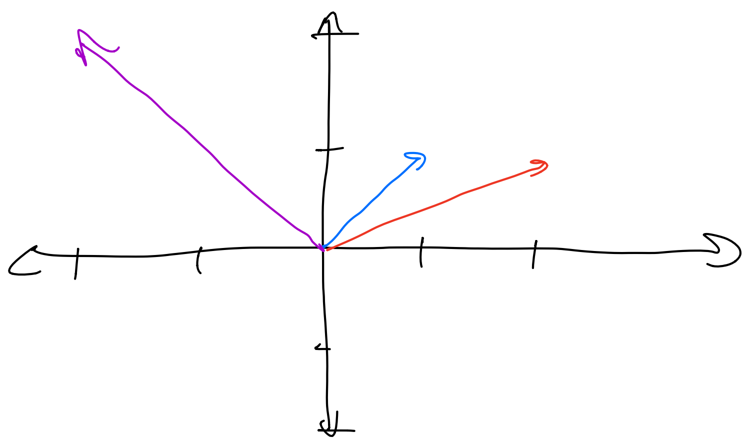
EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 2A

1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.



(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

$$\alpha\vec{x}+\beta\vec{y}=\vec{z}$$

col. Interp

$$\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix} \qquad \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$$

(c) Solve for α, β.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -1 & 4 \end{array}\right] - R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -4 \end{array}\right] \times -1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -4 \end{array}\right] - 2R_1$$

$\alpha = 6$

$\beta = -4$

$$6\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-4)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \checkmark$$

2. Span basics

(a) What is span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

$$\left\{ \vec{v} \;\middle|\; \vec{v} = \alpha \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}$$

$$\vec{v} = \alpha \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \alpha + 2\beta \\ 2\alpha + \beta \\ 0 \end{bmatrix} \sim \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$$

(b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -3 & -5 \\ 0 & 0 & 0 \end{array}\right] -2R_1 \to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{array}\right] /{-3}$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{array}\right] -2R_2$$

↳ can $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ be written as a L.C. of $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

$$\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{5}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(c) What is a possible choice for $\vec{v}$ that would make span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} * \\ * \\ 1 \end{bmatrix}$$

The vector space not in the span of $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is spanned by $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \vec{b} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \vec{b}$$

$b_1 =$ something
$b_2 =$ something
$b_3 = 0$

from (a)

Prove by construction

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 1 & b_2 \\ 0 & 0 & b_3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -3 & b_2 - 2b_1 \\ 0 & 0 & b_3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & \frac{2b_1 - b_2}{3} \\ 0 & 0 & b_3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{2b_2 - b_1}{3} \\ 0 & 1 & \frac{2b_1 - b_2}{3} \\ 0 & 0 & b_3 \end{array}\right] \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \\ \end{array}$$

3. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

(a)

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \operatorname{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

$$\vec{q} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\} = \sum_{i \in [1,N]} a_i \vec{v}_i$$

$$= a_1 \vec{v}_1 + a_2 \vec{v}_2 + \ldots + a_n \vec{v}_n$$

start with defn.

$$= \left(\frac{a_1}{\alpha}\right)(\alpha \vec{v}_1) + a_2 \vec{v}_2 + \ldots + a_n \vec{v}_n$$

transform to get the form (vectors) you want

$$a_1' \triangleq a_1/\alpha \qquad a_2' = a_2 \qquad a_3' = a_3 \quad \ldots$$

variable transform

$$= a_1'(\alpha \vec{v}_1) + a_2' \vec{v}_2 + \ldots + a_n' \vec{v}_n$$

Now we have a linear comb of the RHS vectors

$$\in \operatorname{span}\{\alpha \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

---

for rigor, need to prove in both directions

$$\vec{r} \in \operatorname{span}\{\alpha \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

$$= b_1(\alpha \vec{v}_1) + b_2 \vec{v}_2 + \ldots + b_n \vec{v}_n$$

$$= \alpha b_1 \vec{v}_1 + b_2 \vec{v}_2 + \ldots + b_n \vec{v}_n$$

$$b_1' \triangleq \alpha b_1 \qquad b_{i \neq 1}' = b_i$$

$$\vec{r} = b_1' \vec{v}_1 + b_2' \vec{v}_2 + \ldots + b_n' \vec{v}_n$$

$$\in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

$\Rightarrow \vec{q} \in \text{span}\{\dots\} \quad \vec{q} = \sum c_i \vec{v}_i$

(b) **(Practice)**

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

$$\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$$

$$q = \sum a_i \vec{v}_i$$

$$= a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$$

$$= a_1\vec{v}_1 + \underbrace{a_1\vec{v}_2 - a_1\vec{v}_2}_{=0} + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$$

$$= a_1(\vec{v}_1 + \vec{v}_2) + (a_2 - a_1)\vec{v}_2 + \dots + a_n\vec{v}_n$$

$$a_1' = a_1 \quad a_2' = a_2 - a_1 \quad a_3' = a_3 \quad \dots$$

$$\vec{w}_1 = \vec{v}_1 + \vec{v}_2$$
$$\vec{w}_2 = \vec{v}_2$$
$$\vdots$$

$$\vec{q} = a_1'(\vec{v}_1 + \vec{v}_2) + a_2'\vec{v}_2 + \dots + a_n'\vec{v}_n$$

$$\in \text{span}\{(\vec{v}_1 + \vec{v}_2), \vec{v}_2, \dots, \vec{v}_n\}$$

$$\vec{q} = \sum a_i' \vec{w}_i$$

$$\vec{r} \in \text{span}\{(\vec{v}_1 + \vec{v}_2), \vec{v}_2, \dots, \vec{v}_n\}$$

$$= b_1(\vec{v}_1 + \vec{v}_2) + b_2(\vec{v}_2) + \dots + b_n\vec{v}_n$$

$$= b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \dots + b_n\vec{v}_n$$

$$b_1' = b_1 \quad b_2' = b_1 + b_2 \quad b_3' = b_3 \quad \dots$$

$$\vec{r} = b_1'\vec{v}_1 + b_2'\vec{v}_2 + \dots + b_n'\vec{v}_n = \sum b_i' \vec{v}_i$$

$$\in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$$

Anon. feedback survey tinyurl.com/eecs16a-sp21-bob