Vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

column vector

$N \times 1$

Matrices

Transpose ~ mirror over diagonal

$$X = \begin{bmatrix} x_{11} & x_{12} & & x_{1m} \\ x_{21} & x_{22} & & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & & x_{Nm} \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$N \times M$

matrix as a group of column vectors

$$\vec{y}^T = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix}$$

row vectors

$1 \times N$

$$Y = \begin{bmatrix} — \vec{y}_1^T — \\ \vdots \\ — \vec{y}_m^T — \end{bmatrix}$$

matrix as a stack of row vectors

Linear operations

① Addition

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

2×1 $\quad$ 2×1 $\quad$ 2×1

② Scalar Multiplication

$$\alpha \vec{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} \qquad \beta \vec{y} = \begin{bmatrix} \beta y_1 \\ \beta y_2 \end{bmatrix}$$

Multiplications

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

① Vector Vector

1.1 inner product aka dot product

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{\underline{\vec{x}^T \vec{y}}} = \underset{1 \times 2}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} x_1 y_1 + x_2 y_2 \end{bmatrix}}$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \sum_{i=1}^{N} x_i y_i$$

1.2 outer product $\vec{y}\,\vec{x}^T = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_2 y_1 \\ x_1 y_2 & x_2 y_2 \end{bmatrix}$

$2\times 1 \quad 1\times 2 \quad 2\times 2$

② Matrix - Vector

$$\underset{M\times N}{\begin{bmatrix} a_{11} & & a_{1N} \\ a_{21} & \cdots & a_{2N} \\ \vdots & & \vdots \\ a_{M1} & & a_{MN} \end{bmatrix}} \underset{N\times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}} = \underset{M\times 1}{\begin{bmatrix} a_{11}x_1 + \ldots & a_{1N}x_N \\ & \vdots \\ a_{M1}x_1 + \ldots & a_{MN}x_N \end{bmatrix}}$$

2.1 Row Perspective

$$\vec{a}_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{iN} \end{bmatrix} \qquad \begin{bmatrix} \text{---} & \vec{a}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \vec{a}_m^T & \text{---} \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1^T x \\ \vdots \\ \vec{a}_m^T x \end{bmatrix}$$

stack of Inner products

2.2 Column Perspective

$$\vec{b}_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{Mi} \end{bmatrix} \qquad \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_N \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \vec{b}_1 x_1 + \vec{b}_2 x_2 + \ldots \vec{b}_N x_N \end{bmatrix} = \sum_{i=1}^{N} x_i \vec{b}_i$$

weighted sum of vectors.
(linear combination)

③ Matrix - Matrix

$$\underset{m\times n}{A} \times \underset{n\times p}{B} = A \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \\ | & | & & | \end{bmatrix}$$

extension of matrix - vector

$$= \begin{bmatrix} a_1^T b_1 & \cdots & a_1^T b_p \\ \vdots & & \vdots \\ a_m^T b_1 & \cdots & a_m^T b_m \end{bmatrix}$$

Properties: $AB \neq BA$ Not commutative

Note, $\vec{x}^T\vec{y} = \vec{y}^T\vec{x}$, but that only shows that inner product is commutative, $\vec{x}\vec{y}^T \neq \vec{y}\vec{x}^T$, so multiplication is still not commutative

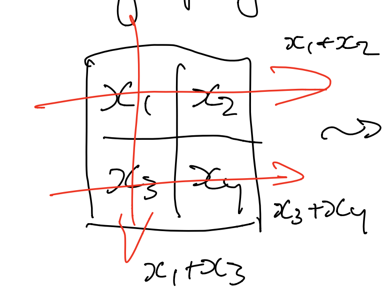
$A(B+C) = AB + AC$ distributive

$A + (B+C) = (A+B) + C$ associative

$A \cdot \vec{0} = \vec{0}$ zero

$A \cdot \vec{I} = A$ identity

Tomography



$\rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\begin{aligned} x_1 + x_2 &= m_1 \\ x_1 + x_3 &= m_2 \\ x_3 + x_4 &= m_3 \end{aligned}$$

linear eqns. describing measurements

$$\begin{bmatrix} \text{———} & \vec{a}_1^T & \text{———} \\ \text{———} & \vec{a}_2^T & \text{———} \\ \text{———} & \vec{a}_3^T & \text{———} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}}_{b}$$

"some set of masks"

"some measurements"

Each row in A describes an image mask. Each measurement m_i is some inner product $a_i^T\vec{x}$. Each element in a_i^T describes how elements in $\vec{x}$ affect the measurement m_i.

EECS 16A Spring 2021 — Designing Information Devices and Systems I

Discussion 1B

1. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

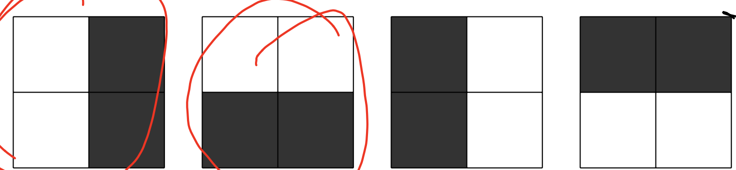
Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

k_1 $\quad$ k_2 $\quad$ mask ~ row

$\vec{k}_1^T \vec{x} = m_1$

Cave Labels

x_1	x_2
x_3	x_4



$x_1 + x_3 = m_1$

$x_1 + x_2 = m_2$

$x_2 + x_4 = m_3$

$x_3 + x_4 = m_4$

(a)

$\vec{k}_2^T \vec{x} = m_2$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}$$

Figure 1: Four image masks.

(a) Let $\vec{x}$ be the four-element vector that represents the magnitude of light emanating from the four cave entrances. Write a matrix $\mathbf{K}$ that performs the masking process in Figure 1 on the vector $\vec{x}$, such that $\mathbf{K}\vec{x}$ is the result of the four measurements.

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

$\frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4 = m_5$

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

see below

(b)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 1 & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \rightarrow \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \end{array} \begin{array}{l} \\ -R_1 \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \begin{array}{l} \\ \\ -R_2 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2 - m_1 \\ 0 & 0 & 1 & 1 & m_3 - m_2 + m_1 \\ 0 & 0 & 0 & 0 & m_4 - m_3 + m_2 - m_1 \end{array}\right] \begin{array}{l} \\ \\ \\ -R_3 \end{array}$$

4 unknowns, 3 pivots $\rightarrow$ No unique soln

RHS = 0 inf.

RHS $\neq$ 0 no soln

(c)

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ \frac{1}{2} & 1 & 1 & \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{array}{l} \\ -R_1 \\ \\ \\ -\frac{1}{2}R_1 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{array}{l} \\ \\ -R_2 \\ \\ -R_2 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ \\ \\ -R_3 \\ -\frac{3}{2}R_3 \end{array}$$

4 unknowns, 4 pivots,
we can find a solution,
$\rightarrow$ if $R_4 \sim 0=0$, then
exact soln.

Linear Independence

A set of vectors $\{\vec{v}_1 \dots \vec{v}_n\}$ is linearly independent if the only soln. for $\sum_{i=1}^{N} c_i \vec{v}_i = \vec{0}$ is $c_i = 0$.

$$c_1 v_1 + c_2 v_2 \dots + c_N v_N = 0$$

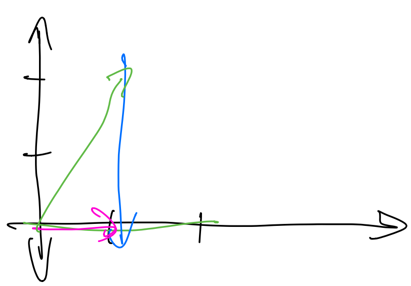
$$c_1 = c_2 = c_3 = \dots = c_N = 0$$

Linearly Dependent

A set of vectors $\{\vec{v}_1 \dots \vec{v}_N\}$ is linealy dependent if there exist some nonzero c_i such that $\sum_{i=1}^{N} c_i \vec{v}_i = 0$

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_{N-1} \vec{v}_N = -c_N \vec{v}_N$$

$\Rightarrow$ you can write one vector as a nonzero sum of the other vectors,



$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ \quad =$$

The 3rd vector can be written as a sum of the first two. So all 3 vectors as a set are linearly dependent. If you remove any one of them, the set becomes linearly independent.

2. Proofs

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is **linearly dependent** if there exists constants $c_1, c_2, \dots c_n$ such that $\sum_{i=1}^{i=n} c_i \vec{v}_i = \vec{0}$ and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

if then

(b) For $\mathbf{A} \in \mathbb{R}^{m \times n}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

if then

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

a) $$A\vec{x} = \vec{0} = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \vec{a}_1 x_1 + \dots + \vec{a}_N x_N = 0$$

$$\Downarrow$$

$$= \sum_{i=1}^{N} x_i \vec{a}_i = 0$$

columns of A are linearly dependent by definition

b) $A\vec{x}_1 = \vec{b}$ $\quad A\vec{x}_2 = \vec{b}$

$$A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b} = 0$$

$$A(\vec{x}_1 - \vec{x}_2) = \vec{0}$$

$\vec{x}_1 \neq \vec{x}_2$

$\vec{x}_1 - \vec{x}_2 \neq 0$

$\vec{x}_3 = \vec{x}_1 - \vec{x}_2 \neq 0$ $\quad A\vec{x}_3 = \vec{0}$

because we proved part (a), this is enough.

c) $A\vec{y} = 0$ $\quad A\vec{x} = \vec{b}$

$\downarrow$

$cA\vec{y} = 0$

$$A\vec{x} + \vec{0} = \vec{b} + \vec{0}$$

$$A\vec{x} + A\vec{y} = \vec{b}$$

$$A(\vec{x} + \vec{y}) = \vec{b}$$ one more soln.

$$A\vec{x} + cA\vec{y} = \vec{b} + \vec{0} = \vec{b}$$

$$A(\vec{x} + c\vec{y}) = \vec{b}$$ inf. solns.

$c = 0, 1, 1.1, \sqrt{2}, \pi, e \ldots$