$x + 2y = 7$
$3x - y = 0$

Algebraic Manipulation/Substitution

① $y = 3x$ Isolate

② $x + 2(3x) = 7$ Substitute

$\Rightarrow x = 1$

③ $y = 3x = 3(1) = 3$:)

Why GE? ① Matrices

② Systematic Method

① $x + 2y = 7$
$3x - y = 0$

Matrix-Vector Form

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 3 & -1 & 0 \end{array}\right]$$

Equivalent steps b/w algebraic manipulation and GE

① Form Augmented Matrix

② Row operations $\rightarrow$ Echelon Form

(1) Scalar Multiply

(2) Addition

② $x + 2y = 7$

$(3x - y) - 3(x + 2y) = 0 - 3(7)$

$0x - 7y = -21$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -7 & -21 \end{array}\right] \begin{array}{l} -3 \quad -6 \quad -21 \\ R_2 \rightarrow R_2 - 3R_1 \end{array}$$

③ $x + 2y = 7$

$0x + 1y = 3$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 3 \end{array}\right] R_2 \rightarrow \tfrac{1}{7}R_2$$

Echelon form $\rightarrow$ values to the left and below each pivot $= \emptyset$

Pivot = leftmost nonzero entry in row

③ Row ops $\rightarrow$ Back substitution to remove nonzero values in each column, aka, reduced echelon form.

④ $x + 2y - 2(1y) = 7 - 2(3)$

$x = 1$

$y = 3$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right] R_1 \rightarrow R_1 - 2R_2$$

$\Rightarrow x = 1$

$y = 3$

① Augmented matrix

② Row ops $\rightarrow$ Echelon form

③ Row ops $\rightarrow$ Back substitution to reduced echelon form

EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 1A

GE steps ① Augmented Matrix ② Row ops to Echelon form ③ Back substitute w/ Row ops.

1. Gaussian Elimination

Use Gaussian elimination to solve the following systems. Does a solution exist? Is it unique?

② Forward phase
1) 1 in pivot
2) subtract down

(a) ①
$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 6 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right] \tfrac{1}{2}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & -2 & 0 \end{array}\right] -R_1$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -6 & 6 \end{array}\right] -2R_2$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \tfrac{1}{6}$$

③ Back sub

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} -2R_3 \\ -2R_3 \\ \end{array}$$

$x_1 = 5$
$x_2 = -1$
$x_3 = -1$

(b)
$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & 3 & 5 & 3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & -2 & 6 & -2 \\ 0 & -1 & 3 & 1 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & 1 \end{array}\right] -\tfrac{1}{2}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right] +R_2$$

$0x_1 + 0x_2 + 0x_3 = 2$

$0 = 2$

No Solution

(c)
$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -6 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + \tfrac{1}{2}x_3 = \tfrac{1}{2}$
$x_2 + x_3 = 3$ infinite soln

$x_3 = 3$ $x_1 = -1$
$x_2 = 0$

(d) True or False: A system of equations with more equations than unknowns will always have either infinite solutions or no solutions.

False:
example w/ 4 eqns

$x_1 = 1$
$x_2 = 2$
$x_3 = 3$
$x_1 + x_3 = 4$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Exact soln exists. Check the number of pivots.

For N unknowns, need N pivots.

(e) **(Practice)**

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & 2/3 & 1/3 \\ 0 & 0 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & 2/3 & 1/3 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

2 pivots, 3 unknowns

Inf soln

$x_1 - 1/3\, x_2 = 0$

$x_3 = 1/2$

Ex: $x_1 = 0$, $x_2 = 0$, $x_3 = 1/2$

(f) **(Practice)**

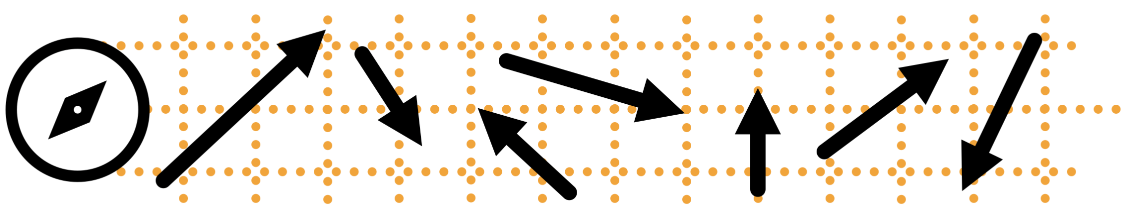
$$\left[\begin{array}{ccccccc} 2x & + & 4y & + & 2z & = & 8 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & 8 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & -3 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -2 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Exact soln $x_1 = 5$ $x_2 = -2$ $x_3 = 3$

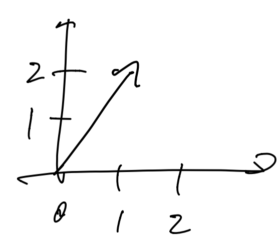
2. Vectors



A vector is an ordered list of numbers. For instance, a point on a plane (x,y) is a vector! We label vectors using an arrow overhead $\vec{v}$, and since vectors can live in ANY dimension of space we'll need to leave our notation general $(x,y) \rightarrow \boxed{\vec{v} =} (v_1, v_2, \ldots)$. Below are few more examples (the left-most form is the general definition):

$(x, y) = (1, 2)$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \vec{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3 \qquad \vec{b} = \begin{bmatrix} 2.4 \\ 5.3 \end{bmatrix} \in \mathbb{R}^2$$



Just to unpack this a bit more, $\vec{b} \in \mathbb{R}^3$ in english means "vector $\vec{b}$ lives in 3-Dimensional space".

- The $\in$ symbol literally means "in"
- The $\mathbb{R}$ stands for "real numbers" (FUN FACT: $\mathbb{Z}$ means "integers" like $-2, 4, 0, \ldots$)

↳ hand write as $\mathbb{R}$

- The exponent $\mathbb{R}^n \leftarrow$ indicates the dimension of space, or the amount of numbers in the vector.

One last thing: it is standard to write vectors in column-form, like seen with $\vec{a}, \vec{b}, \vec{x}$ above. We call these *column vectors*, in contrast to horizontally written vectors which we call *row vectors*.

Okay, let's dig into a few examples:

(a) Which of the following vectors live in $\mathbb{R}^2$ space?

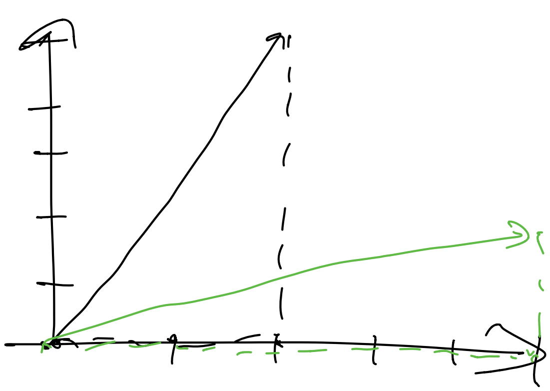
$$i. \begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} \qquad ii. \begin{bmatrix} 5 \\ 0 \\ 3 \\ 5 \end{bmatrix} \qquad iii. \begin{bmatrix} -4.76 \\ 1.32 \\ 0.01 \end{bmatrix} \qquad iv. \begin{bmatrix} -20 \\ 100 \end{bmatrix}$$

$\mathbb{R}^2$ $\qquad$ $\mathbb{R}^4$ $\qquad$ $\mathbb{R}^3$ $\qquad$ $\mathbb{R}^2$

What about this? $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\mathbb{R}^3$

(b) Graphically show the vectors (either in a sketch with axes, or a plot on a computer):

$$i. \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad ii. \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$



(c) Compute the sum $\vec{a}+\vec{b}=\vec{c}$ from the vectors below, and then graphically sketch or plot these vectors. (show them in a way that forms a triangle; also is there only one possible triangle?)

$$\vec{a}=\begin{bmatrix}1\\2\end{bmatrix} \qquad \vec{b}=\begin{bmatrix}4\\3\end{bmatrix}$$

General Vector Addition

$$\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}a+c\\b+d\end{bmatrix}$$

$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}1+4\\2+3\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}$$