Designing Information Systems + Devices

Module 1 / 3

~ Robotics + System Eng.
106, 128, 149

~ Information, Data, ML
126, 127, 188, 189,
Data 100, CS70

~ Signal + Communication
120, 123, 121

Module 2

~ IC engineering
105, 140, 151, 130

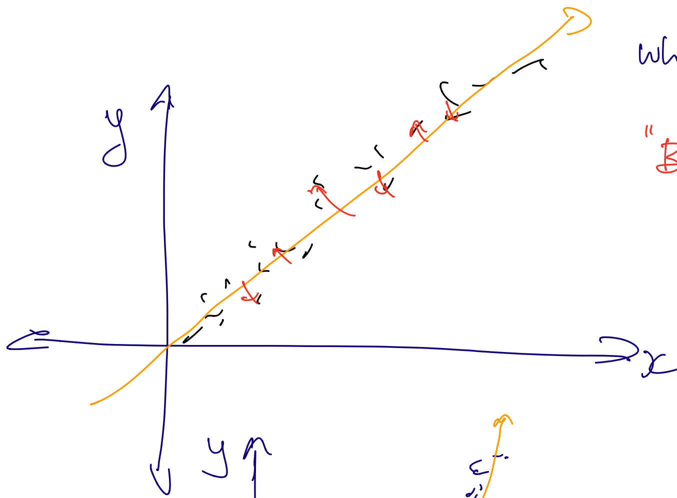
~ Power
113, 105, 130

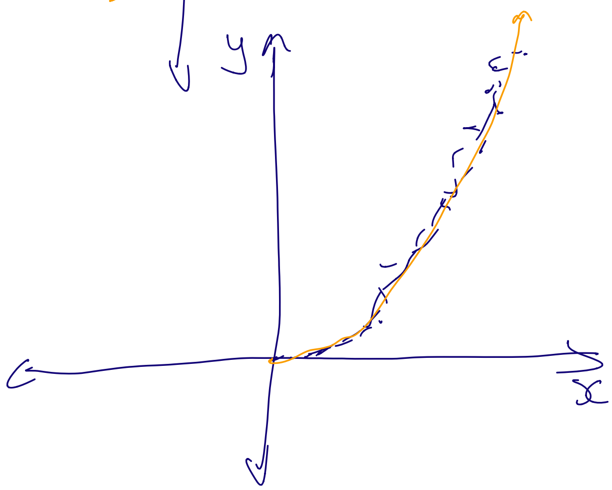
~ optics, LiDAR
117, 118, 130

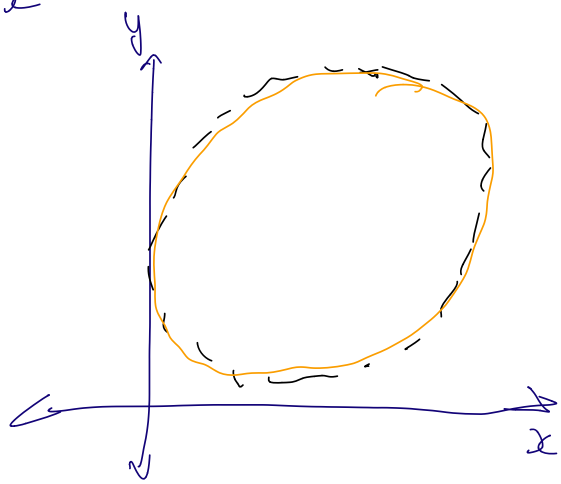
Before all this. Take EECS16B

what is modelling + regression?

"Best fit" → minimize distance of line to points.







$x \longrightarrow$ [Black box] $\longrightarrow y$

try to model what's inside, just like our imaging problem

Recall imaging

$$\begin{bmatrix} \text{masks} \end{bmatrix} \underbrace{\begin{bmatrix} \end{bmatrix}}_{\text{image}} = \begin{bmatrix} \end{bmatrix} \leftarrow \text{measurements}$$

use "x" values to create "masks"

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1. Polynomial Fitting

Let's try an example. Say we know that the output, y, is a quartic polynomial in x. This means that we know that y and x are related as follows:

$$\boxed{y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}$$

Some coeffs. some terms with x.

We're also given the following observations:

x^4	x^3	x^2	x	y
0	0	0	0.0	24.0
0.5^4	0.125	0.25	0.5	6.61
1^4	1	1	1.0	0.0
1.5^4	2.25	1.25	1.5	−0.95
2^4	8	4	2.0	0.07
		6.25	2.5	0.73
		9.0	3.0	−0.12
			3.5	−0.83
			4.0	−0.04
			4.5	6.42

$$x \to \boxed{f(x)} \to y$$

(a) What are the unknowns in this question? What are we trying to solve for?

$$\underbrace{a_0,\ a_1,\ a_2,\ a_3,\ a_4}_{\text{these will describe the model.}}$$

(b) Can you write an equation corresponding to the first observation (x_0, y_0), in terms of a_0, a_1, a_2, a_3, and a_4? What does this equation look like? Is it linear in the unknowns?

$(x_0, y_0) = (0.0, 24.0)$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$
$$= 1 \cdot a_0 + x^1 a_1 + x^2 a_2 + x^3 a_3 + x^4 a_4$$
$$\boxed{24.0 = 1 \cdot a_0 + 0.0^1 a_1 + 0.0^2 a_2 + 0.0^3 a_3 + 0.0^4 a_4}$$

yes, linear

(c) Now, write a system of equations in terms of a_0, a_1, a_2, a_3, and a_4 using *all of the observations*.

$(0.5, 6.61)$ $\quad 6.61 = 1 \cdot a_0 + 0.5^1 a_1 + 0.5^2 a_2 + 0.5^3 a_3 + 0.5^4 a_4$

$(1.0, 0.0)$ $\quad 0.0 = 1 \cdot a_0 + 1^1 a_1 + 1^2 a_2 + 1^3 a_3 + 1^4 a_4$

⋮

$$\boxed{24.0 = 1 \cdot a_0 + 0.0^1 a_1 + 0.0^2 a_2 + 0.0^3 a_3 + 0.0^4 a_4}$$

$$\begin{matrix} 24.0 \\ 6.61 \\ 0.0 \\ \vdots \end{matrix} \begin{matrix} = \\ = \\ = \\ \end{matrix} \begin{matrix} [1 & 0.0^1 & 0.0^2 & 0.0^3 & 0.0^4] \\ [1 & 0.5^1 & 0.5^2 & 0.5^3 & 0.5^4] \\ [1 & 1^1 & 1^2 & 1^3 & 1^4] \\ & \vdots & & & \end{matrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

measurements

Masks / "Data matrix"

image/model

see Python notebook

Proof from lecture

$$\text{Null}(A) = \text{Null}(A^TA)$$

Step 1: if $\vec{v} \in \text{Null}(A)$, $\vec{v} \in \text{Null}(A^TA)$

$$A\vec{v} = \vec{0} \quad \text{for} \quad \vec{v} \neq \vec{0}$$

$$A^TA\vec{v} = A^T\vec{0}$$

$$(A^TA)\vec{v} = \vec{0} \quad \vec{v} \neq 0 \Rightarrow \vec{v} \in \text{Null}(A^TA)$$

Step 2: if $\vec{v} \in \text{Null}(A^TA)$, $\vec{v} \in \text{Null}(A)$

$$A^TA\vec{v} = \vec{0}$$

$$\|A\vec{v}\|^2 = (A\vec{v})^T A\vec{v}$$

$$= v^T A^T A\vec{v}$$

$$= v^T\vec{0}$$

$$\|A\vec{v}\|^2 = 0 \Rightarrow A\vec{v} = \vec{0}$$

$$\Rightarrow \vec{v} \in \text{Null}(A)$$

useful for getting "0"

new trick: replace nullspace terms using defn.

magnitude of vector only $= 0$ if vector $= \vec{0}$

(d) Finally, solve for a_0, a_1, a_2, a_3, and a_4 using IPython. You have now found the quartic polynomial that best fits the data!

2. Orthogonal Subspaces

Two vectors are $\vec{x}$ and $\vec{y}$ are said to be orthogonal if their inner product is zero. That is $\langle \vec{x}, \vec{y} \rangle = 0$. $= x^T y \Rightarrow x \perp y$

Two subspaces $\mathbb{S}_1$ and $\mathbb{S}_2$ of $\mathbb{R}^N$ are said to be orthogonal if all vectors in $\mathbb{S}_1$ are orthogonal to all vectors in $\mathbb{S}_2$. That is,

$$\langle \vec{v}_1, \vec{v}_2 \rangle = 0 \; \forall \vec{v}_1 \in \mathbb{S}_1, \vec{v}_2 \in \mathbb{S}_2.$$

(a) Recall that the *column space* of an $M \times N$ matrix $\mathbf{A}$ is the subspace spanned by the columns of $\mathbf{A}$ and that the *null space* of $\mathbf{A}$ is the subspace of all vectors $\vec{v}$ such that $\mathbf{A}\vec{v} = \vec{0}$.
Prove that the column space of $\mathbf{A}^T$ and null space of any matrix $\mathbf{A}$ are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}^T) \perp \text{Null}(\mathbf{A}) \; \forall \mathbf{A} \in \mathbb{R}^{M \times N}$.
Hint: Use the row interpretation of matrix multiplication.

Known: $\text{Col}(A^T) = \{ b_i \mid A^T x_i = b_i, \; x_i \neq \vec{0} \}$

$\text{Null}(A) = \{ v_i \mid A v_i = \vec{0}, \; v_i \neq \vec{0} \}$

Goal: $b_i \perp v_i \Rightarrow b_i^T v_i = 0$

(b) Now prove that the column space and null space of $\mathbf{A}^T$ of any matrix $\mathbf{A}$ are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}) \perp \text{Null}(\mathbf{A}^T) \; \forall \mathbf{A} \in \mathbb{R}^{M \times N}$.

$$A = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix} \qquad \text{Null}(A) = A v_i = \begin{bmatrix} a_1^T v_i \\ a_2^T v_i \\ \vdots \\ a_n^T v_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

$$A^T = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad \text{Col}(A^T) = A^T x_i = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots x_n \vec{a}_n$$

$$b_i^T v_i = (A^T x_i)^T v_i$$
$$= (x_1 \vec{a}_1 + \ldots + x_n \vec{a}_n)^T v_i$$
$$= (x_1 \vec{a}_1^T + \ldots + x_n \vec{a}_n^T) v_i$$

$$= x_1 \vec{a}_1^T v_i + \ldots + x_n \vec{a}_n^T v_i$$
$$= x_1 0 + x_2 \cdot 0 + \ldots + x \cdot 0 = 0$$

$$b_i^T v_i = 0 \Rightarrow b_i \perp v_i$$

$$\Rightarrow Col(A^T) \perp Null(A)$$

Alt: $b_i^T v_i = (A^T x_i)^T v_i$

$Null(A) = v_i \mid Av_i = \vec{0}$

$= x_i^T A v_i$ replace w/ $\vec{0}$ from Null space Defn.

$= x_i^T \vec{0}$

$b_i^T v_i = 0$ use to get "0"

$\Rightarrow b_i \perp v_i$

$Col(A^T) \perp Null(A)$

(b) Prove $Col(A) \perp Null(A^T)$

Defn $B = A^T$

We know $Col(B^T) \perp Null(B)$

Plug in $B = A^T$

$\Rightarrow Col((A^T)^T) \perp Null(A^T)$

$\Rightarrow Col(A) \perp Null(A^T)$ ✓ yay elephants.