EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 12B

1. Mechanical Projection

In $\mathbb{R}^n$, the vector valued projection of vector $\vec{b}$ onto vector $\vec{a}$ is defined as:

$$\text{proj}_{\vec{a}}\left(\vec{b}\right) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\|\vec{a}\|^2}\vec{a}.$$

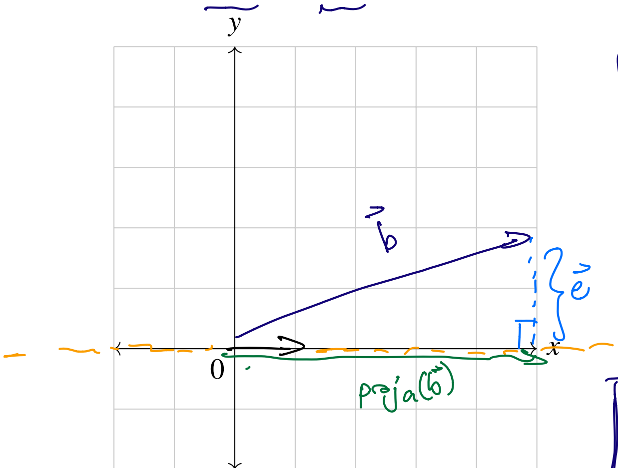
proj$_a$(b) is the vector in span(a) that is "closest" to b.

⇝ $\min \|e\| = \min \| b - \text{proj}_a(b) \|$

happens when $(b - \text{proj}_a(b)) \perp \text{proj}_a(b)$

Recall $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle$.

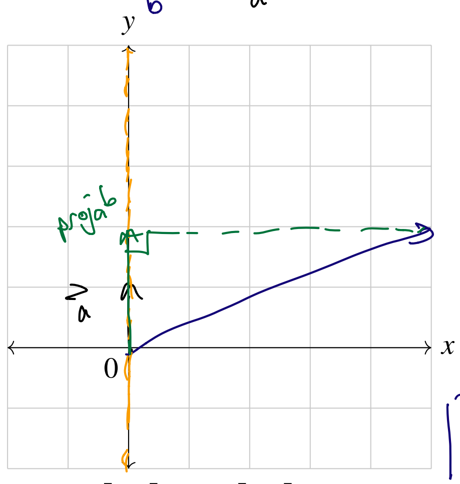
(a) Project $\overset{\vec b}{\begin{bmatrix}5\\2\end{bmatrix}}$ onto $\overset{\vec a}{\begin{bmatrix}1\\0\end{bmatrix}}$ – that is, onto the x-axis. Graph these two vectors and the projection.



$$\text{proj}_{\vec a}(\vec b) = \frac{\langle a, b \rangle}{\langle a, a \rangle}\vec a = \frac{\left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\rangle}\begin{bmatrix}1\\0\end{bmatrix} = \frac{5}{1}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\boxed{\text{proj}_{\vec a}(\vec b) = \begin{bmatrix}5\\0\end{bmatrix}}$$

(b) Project $\underset{\vec b}{\begin{bmatrix}5\\2\end{bmatrix}}$ onto $\underset{\vec a}{\begin{bmatrix}0\\1\end{bmatrix}}$ – that is, onto the y-axis. Graph these two vectors and the projection.



$$\text{proj}_a(b) = \frac{\langle a, b \rangle}{\langle a, a \rangle}\vec a = \frac{\left\langle \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\rangle}\begin{bmatrix}0\\1\end{bmatrix} = \frac{2}{1}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\boxed{\text{proj}_a(b) = \begin{bmatrix}0\\2\end{bmatrix}}$$

(c) Project $\begin{bmatrix}4\\-2\end{bmatrix}$ onto $\begin{bmatrix}2\\-1\end{bmatrix}$. Graph these two vectors and the projection.

$\vec{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

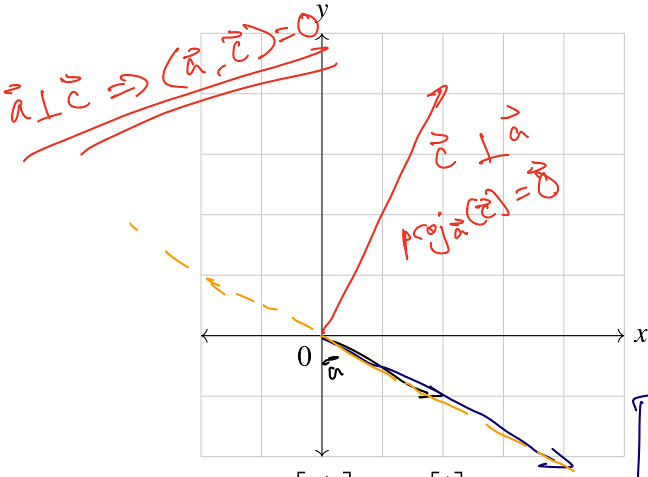
$\vec{a} \parallel \vec{b}$ parallel, $\vec{b} \in \text{span}(\vec{a})$

The closest vector to $\vec{b}$ in span$(\vec{a})$ is $\vec{b}$ itself.

$$\text{proj}_a(\vec{b}) = \frac{\langle a, b \rangle}{\langle a, a \rangle}\vec{a} = \frac{\left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\rangle} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{10}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\boxed{\text{proj}_a(\vec{b}) = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \vec{b}}$$



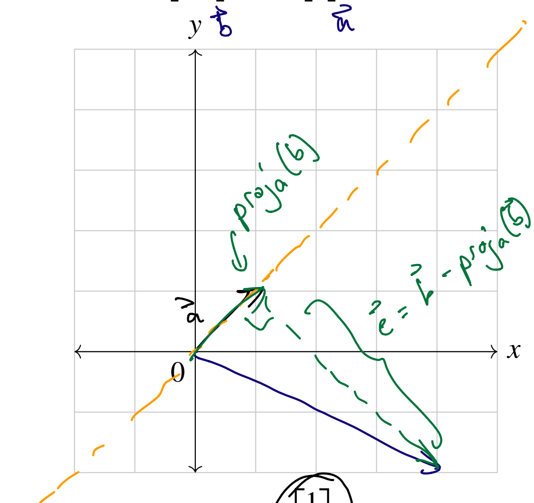
(d) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.

$\vec{b}$ $\quad$ $\vec{a}$

$$\text{proj}_a(\vec{b}) = \frac{\langle a, b \rangle}{\langle a, a \rangle}\vec{a} = \frac{\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{2}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\text{proj}_a(\vec{b}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

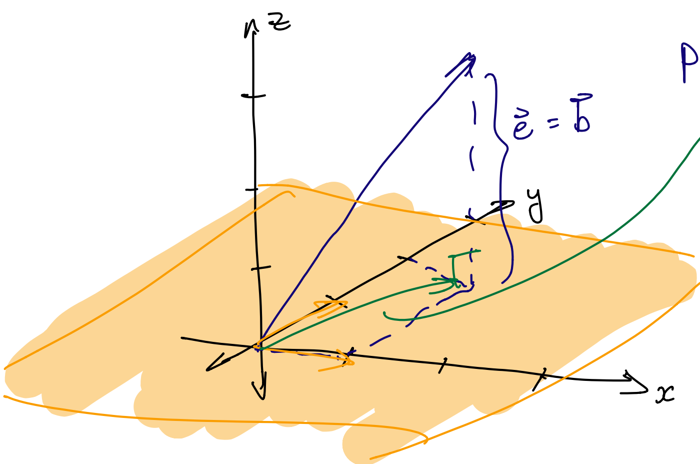


(e) (Practice) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the span of the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ – that is, onto the x-y plane in $\mathbb{R}^3$.

$\leftarrow \vec{b}$

(Hint: From least squares, the matrix $A(A^\top A)^{-1}A^\top$ projects a vector into $C(A)$.)

(f) (Practice) What is the geometric/physical interpretation of projection? Justify using the previous parts.



$\text{proj}_{col(A)}\vec{b} \in \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

get $\hat{b} = \vec{b} - e$ s.t. $\|e\|$ is minimized and $\hat{b} \in \text{span}\{\ \}$

$\min \|\vec{e}\| = \|\vec{b} - \text{proj}_{col(A)}\vec{b}\|$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$col(A) \sim \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\underset{m}{\overset{n}{\begin{bmatrix} A \end{bmatrix}}} \underset{}{\overset{n\times 1}{\begin{bmatrix} \vec{x} \end{bmatrix}}} = \begin{bmatrix} \vec{b} \end{bmatrix}_{m \times 1}$$

$m > n$

$A x = b = \hat{b} + \vec{e}$

$\hat{b} \in Col(A)$

$A\hat{x} = \hat{b}$

$\boxed{\hat{x} = (A^TA)^{-1}A^Tb}$ Least Squares Eqn

$\hat{b} \in Col(A)$

$= A\hat{x} = A(A^TA)^{-1}A^T\vec{b}$

This op. makes $\hat{b}$ close to b and in $Col(A)$.

$\rightarrow \text{proj}_{col(A)}(b) = A(A^TA)^{-1}A^T\vec{b}$

$3\ \left[A\right]_{2} \left[x\right]_{2\times 1} = \left[b\right]_{3\times 1}$

2. Least Squares with Orthogonal Columns

Suppose we would like to solve the least squares problem for $\mathbf{A} \in \mathbb{R}^{3\times 2}$ and $\vec{b} \in \mathbb{R}^3$; that is, find an optimal vector $\vec{x} \in \mathbb{R}^2$ which gets $\mathbf{A}\vec{x}$ closest to $\vec{b}$ such that the distance $||\vec{e}|| = ||\vec{b} - \mathbf{A}\vec{x}||$ is minimized. Call this optimal vector $\hat{\vec{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$. Mathematically, we can express this as:

$$||\vec{b} - \mathbf{A}\hat{\vec{x}}||^2 = \min_{\vec{x}\in\mathbb{R}^2} ||\vec{b} - \mathbf{A}\vec{x}||^2 = \min_{\vec{x}\in\mathbb{R}^2} \left\| \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2.$$

To identify the solution $\hat{\vec{x}}$, we may recall the least squares formula: $\hat{\vec{x}} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\vec{b}$, which is applicable when $\mathbf{A}$ has linearly independent columns. We would now like to walk through the intuition behind this formula for the case when $\mathbf{A}$ has orthogonal columns: $\langle \vec{a}_1, \vec{a}_2 \rangle = 0$.

(a) On the diagram below, please label the following elements:

NOTE: For this sub-part only, the matrix $\mathbf{A}$ does not have orthogonal columns.

span$\{\vec{a}_1, \vec{a}_2\}$ $\quad$ $\mathbf{A}\,\hat{\vec{x}}$ $\quad$ $\hat{x}_1\,\vec{a}_1$ $\quad$ $\hat{x}_2\,\vec{a}_2$ $\quad$ $C(\mathbf{A})$ $\quad$ $\vec{e} = \vec{b} - \mathbf{A}\,\hat{\vec{x}}$ $\quad$ $\text{proj}_{C(\mathbf{A})}(\vec{b})$.

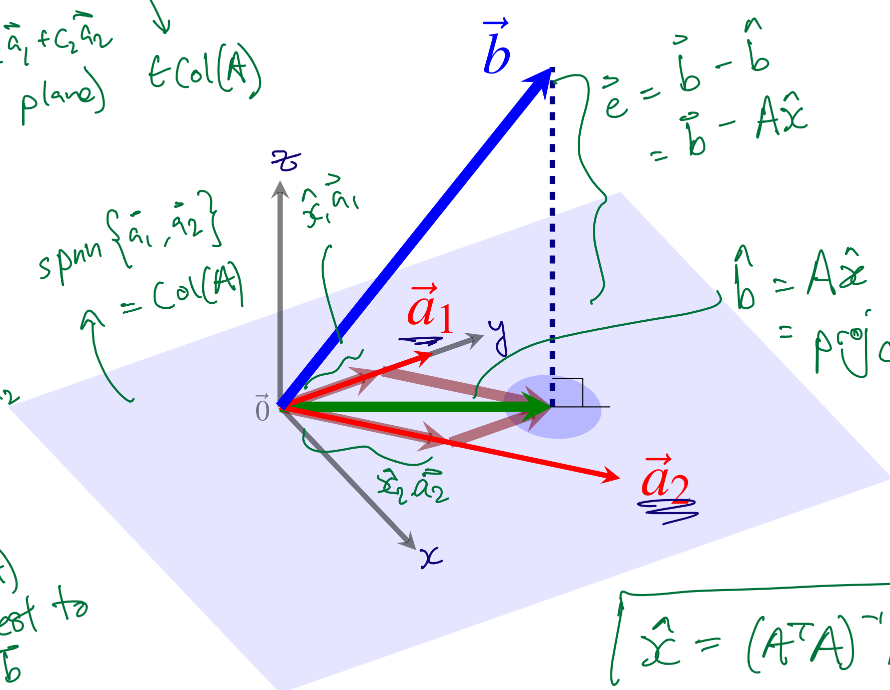
all $c_1\vec{a}_1 + c_2\vec{a}_2$ (2D plane) $\quad$ $\in Col(A)$

$\vec{e} = \vec{b} - \hat{b}$
$= \vec{b} - A\hat{x}$

$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

span$\{\vec{a}_1, \vec{a}_2\}$ $= Col(A)$

$Col(A) = \text{span}\{\vec{a}_1, \vec{a}_2\}$



$\hat{b} = A\hat{x}$
$= \text{proj}_{Col(A)}(\vec{b})$

$A\hat{x} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \hat{x}_1\vec{a}_1 + \hat{x}_2\vec{a}_2$

$A\hat{x} = \hat{b}$ $\quad$ $\hat{b} \in col(A)$ & be closest to $\vec{b}$

$\boxed{\hat{x} = (A^TA)^{-1}A^Tb}$
Least Squares

(b) Now suppose we assume a special case of the least squares problem where the columns of $\mathbf{A}$ are orthogonal (illustrated in the figure below). Given that $\hat{\vec{x}} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\vec{b}$, and $\text{proj}_{C(\mathbf{A})}(\vec{b}) = \mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\vec{b} = \mathbf{A}\hat{\vec{x}}$, show the following statement holds.

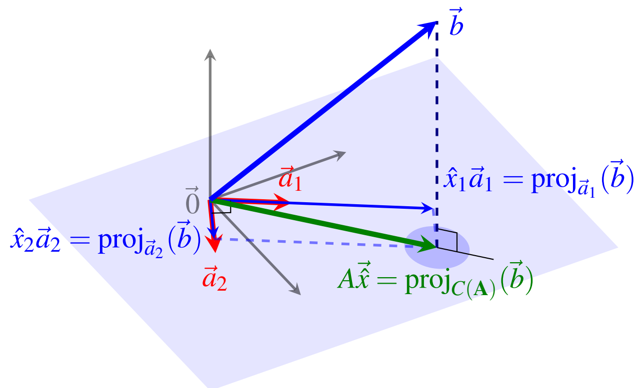
$\hat{b} = \text{proj}_{Col(A)}(b) = A\hat{x} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \frac{\langle a_1, b\rangle}{\langle a_1, a_1\rangle}\vec{a}_1 + \frac{\langle a_2, b\rangle}{\langle a_2, a_2\rangle}\vec{a}_2$

$$\langle \vec{a}_1, \vec{a}_2 \rangle = 0 \implies \boxed{\hat{\vec{x}} = \begin{bmatrix} \frac{\langle \vec{a}_1, \vec{b}\rangle}{||\vec{a}_1||^2} \\ \frac{\langle \vec{a}_2, \vec{b}\rangle}{||\vec{a}_2||^2} \end{bmatrix}} \quad \text{and} \quad \text{proj}_{C(\mathbf{A})}(\vec{b}) = \text{proj}_{\vec{a}_1}(\vec{b}) + \text{proj}_{\vec{a}_2}(\vec{b})$$

if vectors are not orthogonal, you can still use regular least squares,

In words, the statement says that when the columns of $\mathbf{A}$ are orthogonal, the entries of the least squares solution vector $\hat{\vec{x}}$ can be computed by using $\vec{b}$ and only the single other vector $\vec{a}_i$, and that the projection of $\vec{b}$ onto $C(\mathbf{A})$ can be computed by summing the projections of $\vec{b}$ onto the $\vec{a}_i$.

RECALL... $\text{proj}_{\vec{a}_1}(\vec{b}) = \frac{\langle \vec{a}_1, \vec{b} \rangle}{||\vec{a}_1||^2}\vec{a}_1$, $||\vec{a}||^2 = \langle \vec{a}, \vec{a} \rangle$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\mathbf{A} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$



(c) Compute the least squares solution $\vec{\hat{x}} \in \mathbb{R}^2$ to the following system:

$$\min_{\vec{x} \in \mathbb{R}^2} \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2.$$

$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

HINT: Notice that the columns of **A** are orthogonal!!

$A = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \\ a_1 \quad a_2 \end{array}$ $\langle \vec{a}_1, \vec{a}_2 \rangle = 0$ orthogonal

Method 1: Full LS.

$\hat{x} = (A^T A)^{-1} A^T b$

Method 2: Shortcut

$$\hat{x} = \begin{bmatrix} \frac{\langle a_1, b \rangle}{\langle a_1, a_1 \rangle} \\ \frac{\langle a_2, b \rangle}{\langle a_2, a_2 \rangle} \end{bmatrix} = \begin{bmatrix} \frac{1}{1} \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$$

Derivation of orthogonal shortcut.

Assume $a_1 \perp a_2 \Rightarrow \langle a_1, a_2 \rangle$

$$\hat{x} = (A^T A)^{-1} A^T \vec{b} \qquad A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

$$= \left(\begin{bmatrix} — a_1^T — \\ — a_2^T — \end{bmatrix} \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} \right)^{-1} \begin{bmatrix} — a_1^T — \\ — a_2^T — \end{bmatrix} \begin{bmatrix} | \\ b \\ | \end{bmatrix}$$

$$= \left(\begin{bmatrix} a_1^T a_1 & a_1^T a_2 \\ a_2^T a_1 & a_2^T a_2 \end{bmatrix} \right)^{-1} \underbrace{\begin{bmatrix} — a_1^T — \\ — a_2^T — \end{bmatrix} \begin{bmatrix} | \\ b \\ | \end{bmatrix}}$$

$$= \left(\begin{bmatrix} a_1^T a_1 & 0 \\ 0 & a_2^T a_2 \end{bmatrix} \right)^{-1} \left(\cdots \right)$$

$$= \frac{1}{a_1^T a_1 \cdot a_2^T a_2} \begin{bmatrix} a_2^T a_2 & 0 \\ 0 & a_2^T a_2 \end{bmatrix} \left(\cdots \right)$$

$$= \begin{bmatrix} \frac{1}{a_1^T a_1} & 0 \\ 0 & \frac{1}{a_2^T a_2} \end{bmatrix} \begin{bmatrix} — a_1^T — \\ — a_2^T — \end{bmatrix} \begin{bmatrix} | \\ b \\ | \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a_1^T a_1} & 0 \\ 0 & \frac{1}{a_2^T a_2} \end{bmatrix} \begin{bmatrix} a_1^T b \\ a_2^T b \end{bmatrix} = \begin{bmatrix} \frac{a_1^T b}{a_1^T a_1} \\ \frac{a_2^T b}{a_2^T a_2} \end{bmatrix} = \begin{bmatrix} \frac{\langle a_1, b \rangle}{\langle a_1, a_1 \rangle} \\ \frac{\langle a_2, b \rangle}{\langle a_2, a_2 \rangle} \end{bmatrix}$$