EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 11B

Reference: Inner products

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} \in \mathbb{R}^1$

For this course we will use a standard inner product definition from matrix-vector multiplication:

$$\boxed{\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_v ,} \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n.$$

Defn. of Inner product

In general, any inner product $\langle \cdot, \cdot \rangle$ on a real vector space $\mathbb{V}$ is a bilinear function that satisfies the following three properties:

(a) **Symmetry:** $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

(b) **Linearity:** $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$ and $\langle c\vec{x}, \vec{y} \rangle = c\langle \vec{x}, \vec{y} \rangle$, where $c \in \mathbb{R}$ is a real number.

(c) **Non-negativity:** $\boxed{\langle \vec{x}, \vec{x} \rangle \geq 0,}$ with equality if and only if $\vec{x} = \vec{0}$.

Here $\vec{x}$, $\vec{y}$, and $\vec{z}$ can be any vectors in the vector space $\mathbb{V}$.
The norm (or length) of a vector $\vec{x} = [x_1, x_2, \ldots, x_n]^T$ is defined using the inner product as

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} \equiv \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} .$$

length ≥ 0 inner product w/ itself ≥ 0

1. **Inner Product Properties**

For this question we will verify our coordinate definition of the inner product

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_v , \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n \rightarrow \mathbb{R}^2$$

indeed satisfies the key properties required for all inner products, but presently for the 2-dimensional case. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ for the following parts:

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

(a) Show symmetry $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$:

$$\langle \vec{x}, \vec{y} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle = x_1y_1 + x_2y_2$$
$$= y_1x_1 + y_2x_2 = \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = \langle \vec{y}, \vec{x} \rangle \checkmark$$

(b) Show linearity $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$, where $c \in \mathbb{R}$ is a real number.

$$\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle$$

$$\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = \vec{x}^T(c\vec{y} + d\vec{z})$$
$$= c\vec{x}^T\vec{y} + d\vec{x}^T\vec{z}$$
$$= c\langle x, y \rangle + d\langle x, z \rangle$$

$$= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} cy_1 + dz_1 \\ cy_2 + dz_2 \end{bmatrix} \right\rangle = x_1cy_1 + x_1dz_1 + x_2cy_2 + x_2dz_2$$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T\vec{y}$$

$$= c[x_1y_1 + x_2y_2] + d[x_1z_1 + x_2z_2]$$
$$= c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle \checkmark$$

(c) Show non-negativity $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$:

$$\langle \vec{x}, \vec{x} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = x_1^2 + x_2^2 \geq 0 \text{ for any } x_1, x_2$$

≥ 0 for x_2

≥ 0 for any x_1

if $x_1 = 0$, $x_1^2 = 0$

else, $x_1^2 > 0$

if $x_2 = 0$, $x_2^2 = 0$

else, $x_2^2 > 0$

for $\langle \vec{x}, \vec{x} \rangle = 0$

then $x_1 = 0$, $x_2 = 0$

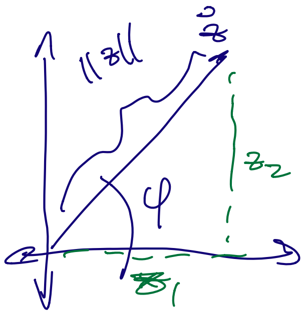
$\Rightarrow \vec{x} = \vec{0}$

2. Geometric Interpretation of the Inner Product ~ similarity

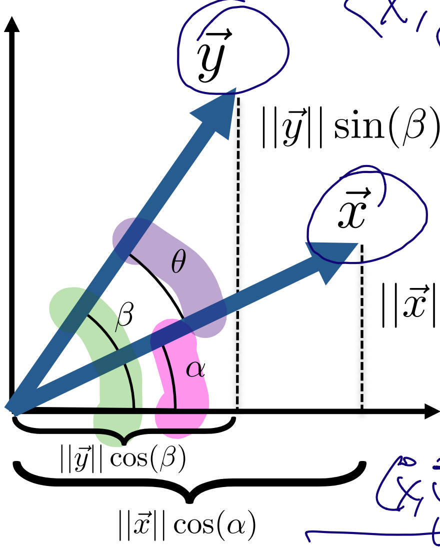
In this problem we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

(a) Derive a formula for the inner product of two vectors in terms of their magnitudes and the angle between them. The figure below may be helpful:



$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \|\vec{z}\|\cos(\varphi) \\ \|\vec{z}\|\sin(\varphi) \end{bmatrix}$



$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2$

$= \|x\|\cos(\alpha)\|y\|\cos(\beta) + \|x\|\sin(\alpha)\|y\|\sin(\beta)$

$= \|x\|\|y\|\left(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\right)$

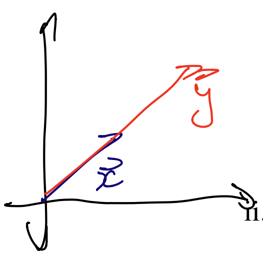
$\langle \vec{x}, \vec{y} \rangle = \|x\|\|y\|\cos(\alpha - \beta)$

$$\boxed{\langle \vec{x}, \vec{y} \rangle = \|x\|\|y\|\cos(\theta)}$$

magnitudes angle difference

(b) For each sub-part, identify any two (nonzero) vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ that satisfy the stated condition and compute their inner product.

i. Identify a pair of parallel vectors: $\theta = 0°$ ⟹ positive inner product "large"

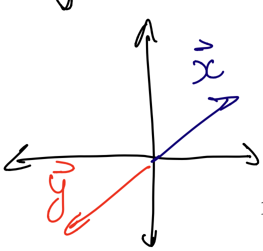


$\vec{x} = \alpha\vec{y}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\langle x, y \rangle = 1 \cdot 2 + 1 \cdot 2$ $\quad \cos(0°) = 1$

$= 4$ positive inner product

ii. Identify a pair of anti-parallel vectors: $\theta = 180°$ ⟹ negative inner product "large"



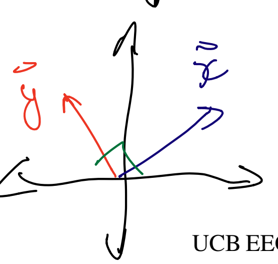
$\vec{x} = -\alpha\vec{y}$ $\quad \alpha > 0$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$\langle x, y \rangle = 1 \cdot -2 + 1 \cdot -2$ $\quad \cos(180°) = -1$

$= -4$ negative inner product

iii. Identify a pair of perpendicular vectors:

$\theta = 90°$ $\quad \cos(90°) = 0$



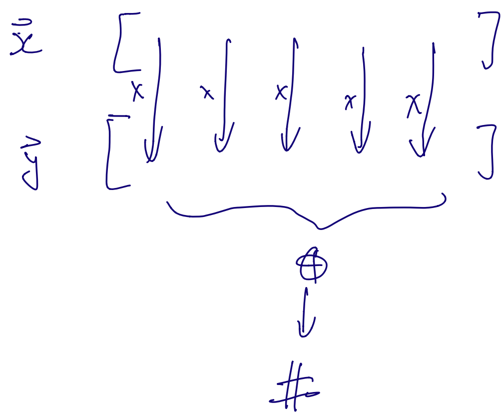
$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \langle x, y \rangle = 1 \cdot -1 + 1 \cdot 1$

$= 0$ $\quad \emptyset$ inner product

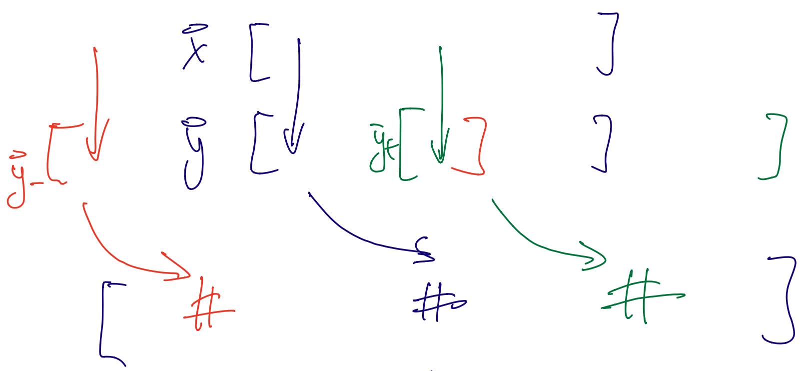
$\|\langle x, y \rangle\| \uparrow$ ~ similar

$\|\langle x, y \rangle\| \to 0$ ~ dissimilar, perpendicular

Inner Product



Cross correlation

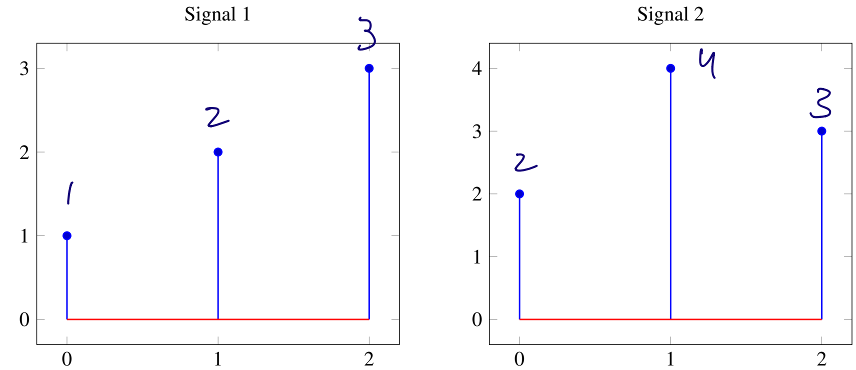


different inner product associated w/ different time shifts of $\vec{y}$ vector

how "similar" $\vec{x}$ is to different time shifts of $\vec{y}$

3. Correlation

We are given the following two signals, $s_1[n]$ and $s_2[n]$ respectively.



Find the cross correlations, $\text{corr}_{s_1}(s_2)$ and $\text{corr}_{s_2}(s_1)$ for signals $s_[n]$ and $s_2[n]$. Recall

$$\boxed{\text{corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k].}$$

$$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$$

sliding $\vec{y}$ to the left ($\vec{y}$ appears earlier)

$\sum_{i=-\infty}^{+\infty} s_1[i]\, s_2[i+2]$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n+2]$	2	4	3	0	0	0	0	
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0 +	0 +	3 +	0 +	0 +	0 +	0 =	$\boxed{3}$

t=0

$\sum_{i=-\infty}^{+\infty} s_1[i]\, s_2[i+1]$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n+1]$	0	2	4	3	0	0	0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0 +	0 +	4 +	6 +	0 +	0 +	0 =	$\boxed{10}$

t=0

$\sum_{i=-\infty}^{+\infty} s_1[i]\, s_2[i]$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n]$	0	0	2	4	3	0	0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0 +	0 +	2 +	8 +	9 +	0 +	0 =	$\boxed{19}$

sliding $\vec{y}$ to the right ($\vec{y}$ appears later)

t=0

$\sum_{i=-\infty}^{+\infty} s_1[i]\, s_2[i-1]$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-1]$	0	0	0	2	4	3	0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0 +	0 +	0 +	4 +	12 +	0 +	0 =	$\boxed{16}$

$\vec{s}_1$	0	0	1	2	3	0	0	
$\vec{s}_2[n-2]$	0	0	0	0	2	4	3	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0 +	0 +	0 +	0 +	6 +	0 +	0 =	$\boxed{6}$

$$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$$

$\text{corr}_{s_2}(s_1)[k]$ by flipping these 2 ...

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n+2]$								
$\langle \vec{s}_2, \vec{s}_1[n+2] \rangle$	+	+	+	+	+	+	=	6

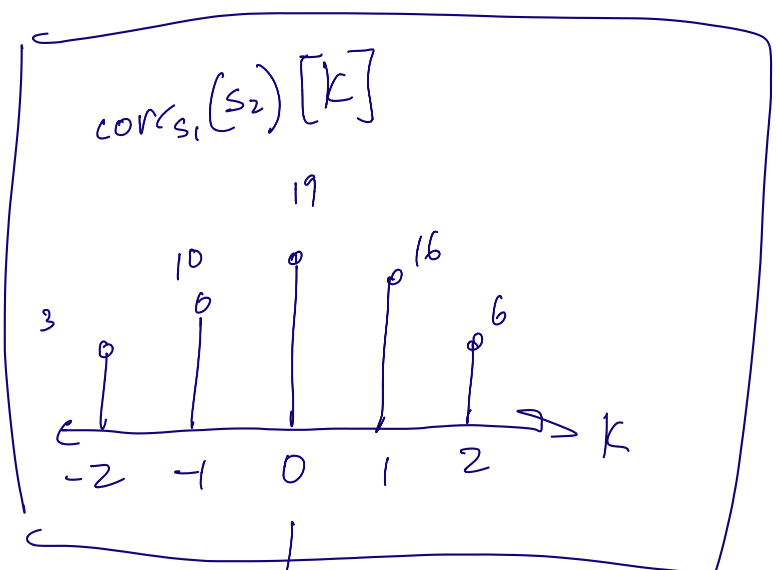
$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n+1]$								
$\langle \vec{s}_2, \vec{s}_1[n+1] \rangle$	+	+	+	+	+	+	=	16

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n]$								
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	+	+	+	+	+	+	=	19

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n-1]$								
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	+	+	+	+	+	+	=	10

$\vec{s}_2$	0	0	2	4	3	0	0	
$\vec{s}_1[n-2]$								
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	+	+	+	+	+	+	=	3

the order flips.



most "similar" when $k=0$ (max $\text{corr}_{s_1}(s_2)$)