EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 12A

1. Search and Rescue Dogs

Berkeley's Puppy Shelter needs your help! While Mr. Muffin was being walked, the volunteer let go of his leash and he is now running wild in the streets of Berkeley (which are quite dangerous)! Thankfully, all of the puppies at the shelter have a collar that sends a bluetooth signal to receiver towers, which are spread throughout the streets (pictured below). If the puppy/collar is within range of the receiver tower, the collar will send the tower a message: the distance of the collar to the tower. Each cell tower has a range of 5 city blocks. Can you help the shelter locate their lost puppy?

Note: A city block is defined as the middle of an intersection to the middle of an adjacent intersection (scale provided on map.) Mr. Muffin is constrained to running wild in the streets, meaning he won't be found in any buildings. If your TA asks 'Where is Mr. Muffin?' it is sufficient to answer with his intersection or 'between these two intersections.'

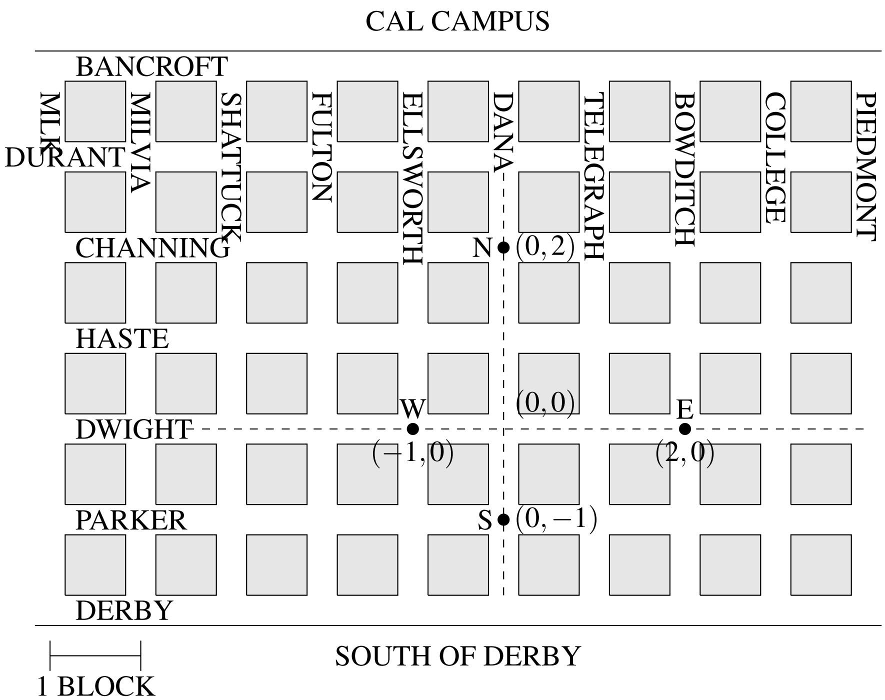

[1]

(a) You check the logs of the cell towers, and they have received the following messages:

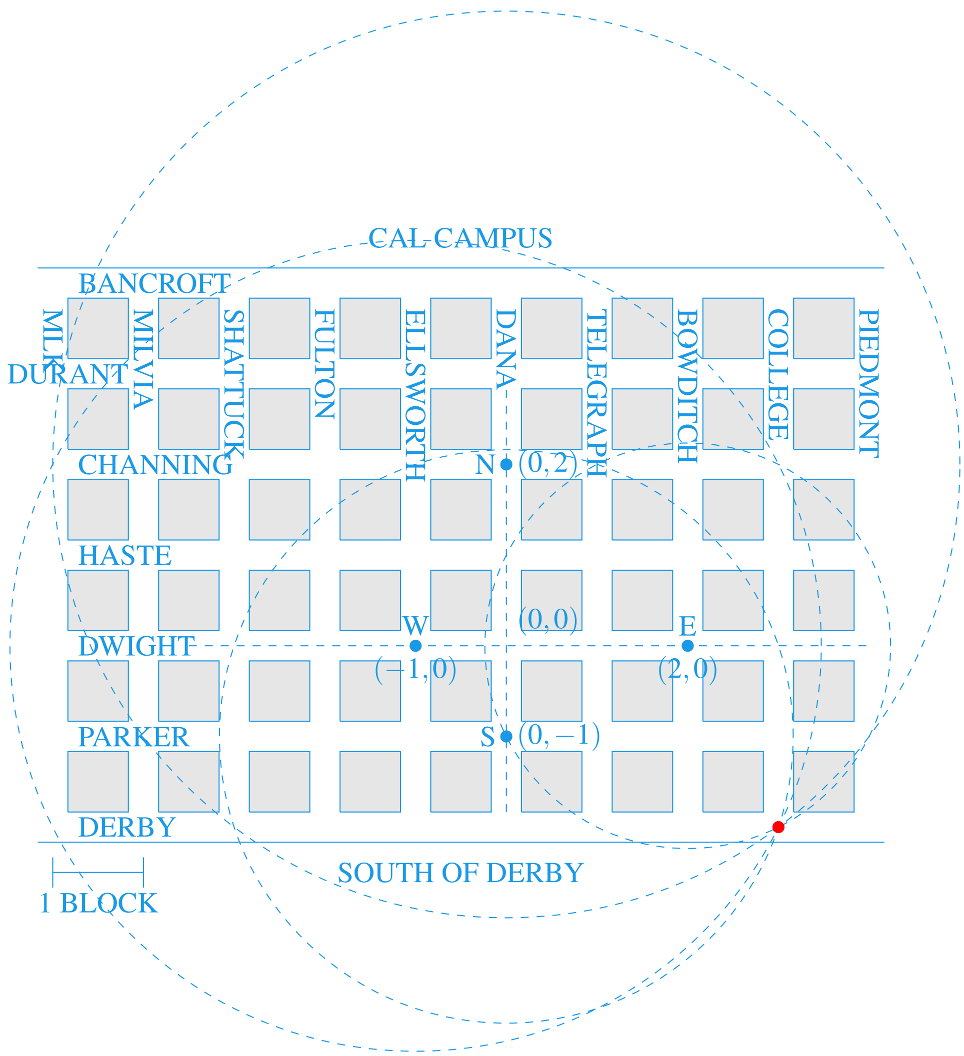
Sensor	Distance
N	5
W	$\sqrt{20}$
E	$\sqrt{5}$
S	$\sqrt{10}$

On the map provided, identify where Mr. Muffin is!

[1] http://www.pupsmile.com/wp-content/uploads/2012/11/running_happy_dog-1024x684.jpeg


CAL CAMPUS
BANCROFT
DURANT
CHANNING
HASTE
DWIGHT
PARKER
DERBY
MLK
MILVIA
SHATTUCK
FULTON
ELLSWORTH
DANA
TELEGRAPH
BOWDITCH
COLLEGE
PIEDMONT
N
(0,2)
(0,0)
W
(−1,0)
E
(2,0)
S
(0,−1)
SOUTH OF DERBY
1 BLOCK

Answer:


CAL CAMPUS
BANCROFT
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CHANNING
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DWIGHT
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DERBY
SOUTH OF DERBY
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COLLEGE
PIEDMONT
N (0,2)
(0,0)
W
(−1,0)
E
(2,0)
S (0,−1)
1 BLOCK

(b) Can you set this up as a system of equations? Are these equations linear? If not, can these equations be linearized? If you can linearize these equations, **write down a simplified form of your set of equations.**

Hint: Set $(0,0)$ to be Dwight and Dana.

Hint 2: Distance $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Hint 3: You don't need all 4 equations. You have two unknowns, x and y. You know from lecture that you need three circles to uniquely find a point. How can you use the third circle/equation to get two equations and two unknowns?

Note: Remember to check for consistency for all nonlinear equations after finding the coordinates.

Answer:

First, set up the system of equations:

$$(x-0)^2 + (y-2)^2 = 5^2$$
$$(x+1)^2 + (y-0)^2 = \sqrt{20}^2$$
$$(x-2)^2 + (y-0)^2 = \sqrt{5}^2$$

Simplify out:

$$x^2 + y^2 - 4y + 4 = 25$$
$$x^2 + 2x + 1 + y^2 = 20$$
$$x^2 - 4x + 4 + y^2 = 5$$

Then subtract equation (1) from equations (2) and (3):

$$2x + 4y - 3 = 20 - 25$$
$$-4x + 4y = 5 - 25$$

This solves to $x = 3, y = -2$ which is roughly College and Derby.

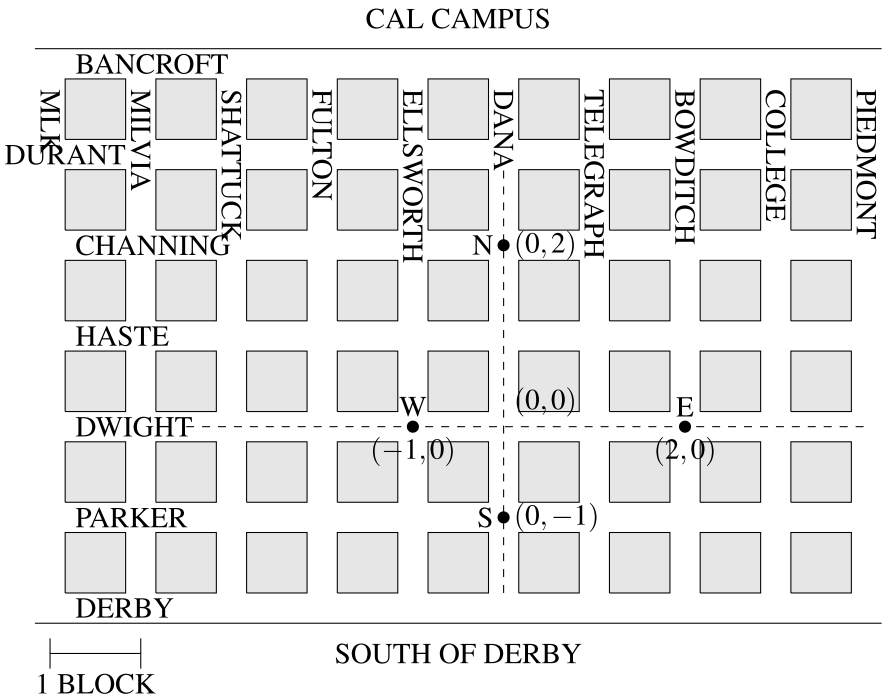
Note that we only need three out of four of the equations to find the location of Mr. Muffin and could omit any one of the four. We can verify that this solution is valid with our fourth equation:

$$(x-0)^2 + (y+1)^2 = \sqrt{10}^2$$
$$(3-0)^2 + (-2+1)^2 = 10$$

(c) Suppose Mr. Muffin is moving fast, and by the time you get to destination in part (a) he's already run off! You check the logs of the cell towers again, and see the following updated messages:

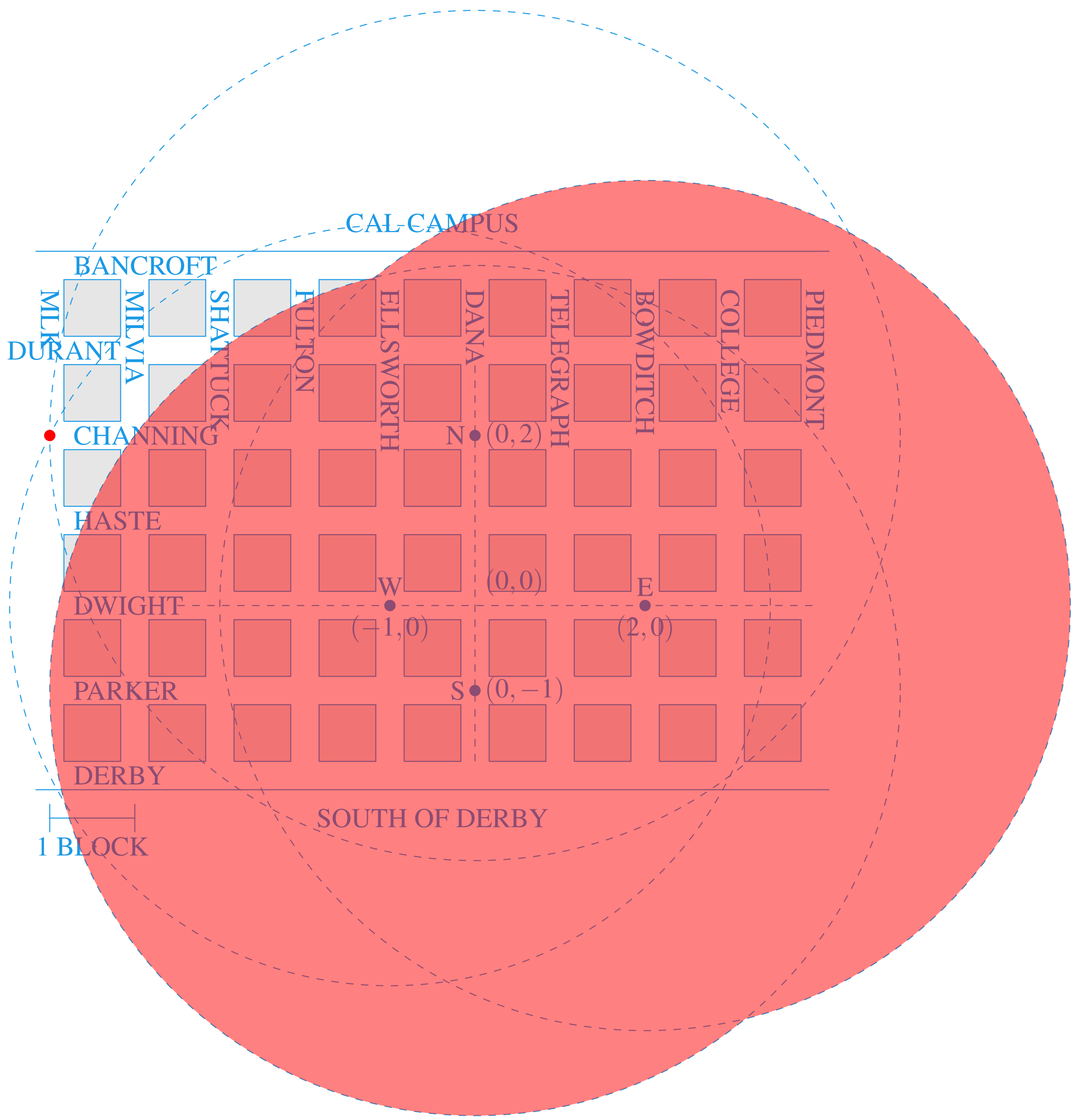
Sensor	Distance
N	5
W	$\sqrt{20}$
E	Out of Range
S	Out of Range

Can you find Mr. Muffin? If so, on the map provided, identify where Mr. Muffin is! (*Note:* Each cell tower has a range of 5 city blocks)



Answer:

With two out of range sensors, you might think that you will not be able to find a unique solution (you need 3 circles to intersect at a point.) The trick is that out of range still provide information on where Mr. Muffin is NOT. See the diagram below - Mr. Muffin cannot be in the shaded region. Therefore, Mr. Muffin should be in Channing and MLK.


CAL CAMPUS
BANCROFT
DURANT
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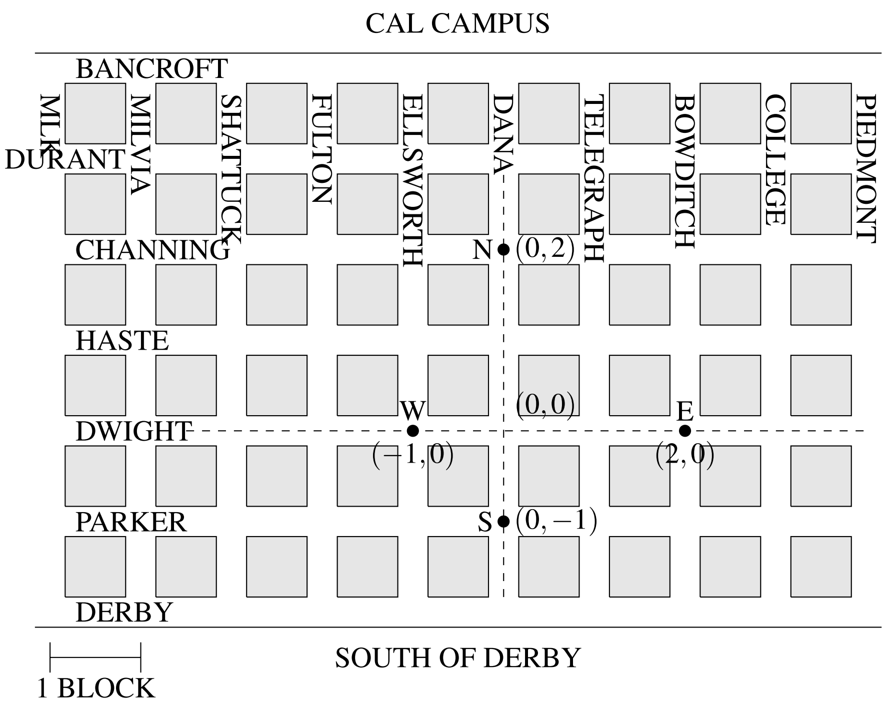
TELEGRAPH
BOWDITCH
COLLEGE
PIEDMONT
N (0,2)
(0,0)
W
(−1,0)
E
(2,0)
S (0,−1)
1 BLOCK

(d) Mr. Muffin is a very mischievous puppy, and while playing and running around he damaged his collar. The transmitter on his collar will still send a signal to the receiver towers, but the distance sensor has noise. You check the logs of the cell towers, and they have received the following messages:

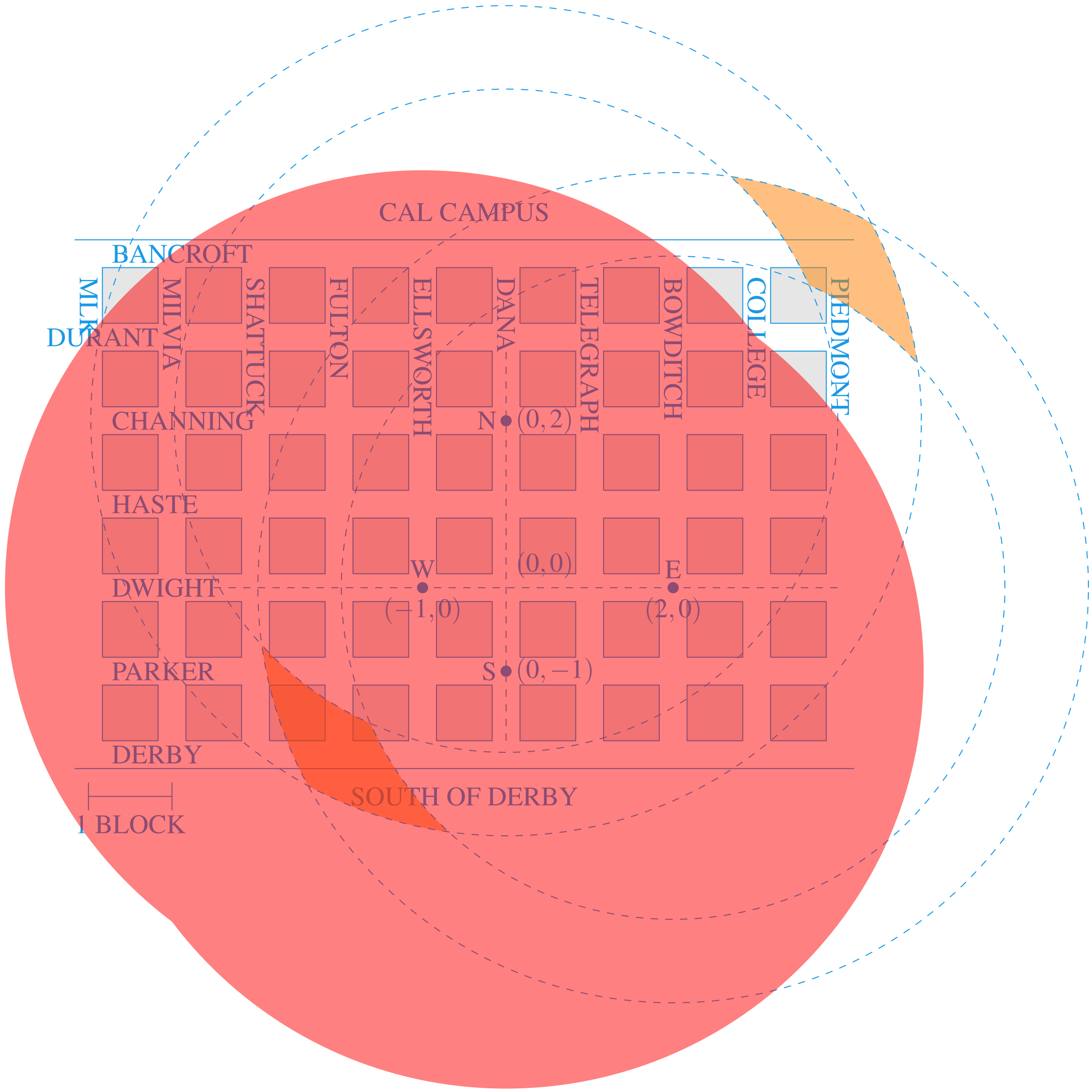
Sensor	Distance
N	4.5 ± 0.5
W	Out of Range
E	4.5 ± 0.5
S	Out of Range

On the map provided, identify where Mr. Muffin is! Can you find exactly where he is? (*Note:* Each cell tower has a range of 5 city blocks)



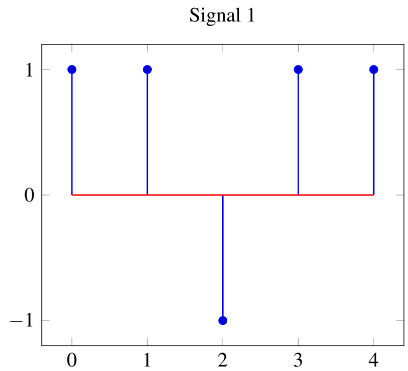
Answer:

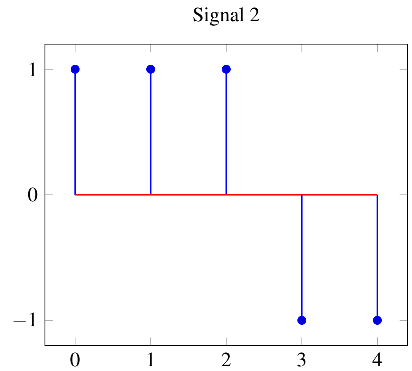
You can't find exactly where he is, but you know he is somewhere between Piedmont/College and Bancroft. See the diagram below.



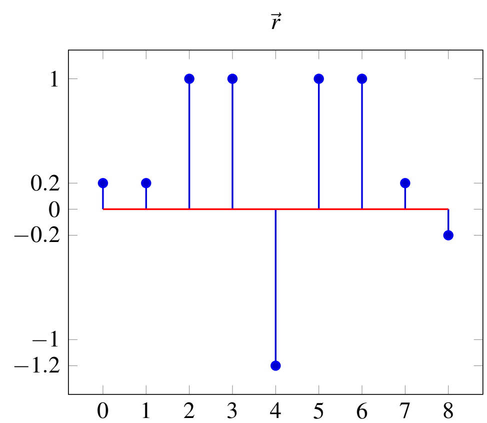
2. Identifying satellites and their delays

We are given the following two signals, $\vec{s}_1$ and $\vec{s}_2$ respectively, that are signatures for two satellites. Your cell phone recieves signals from these two satellites and given a recieved signal $r[n]$ you can identify which, if any, satellite sent the message based on their personal codes.





(a) Your cellphone antenna receives the following signal $r[n]$. You know that there may be some noise present in $r[n]$ in addition to the transmission from the satellite.



By computing the cross-correlations, can you identify which satellite(s) most likely sent the signal, and by what shift the code is identified relative to our received signal? You can use iPython to compute the cross-correlation. When using iPython to plot, think about the range of shifts k that we are interested in plotting based on the lengths of the signals.

Answer: We calculate both $\text{corr}_r(\vec{s}_1)[k]$ and $\text{corr}_r(\vec{s}_2)[k]$:

$\text{corr}_{\vec{r}}(\vec{s}_1)[k]$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n+4]$	1		0		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+4] \rangle$	0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0.2

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n+3]$	1		1		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+3] \rangle$	0.2	+	0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0.4

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n+2]$	−1		1		1		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_1[n+2] \rangle$	−0.2	+	0.2	+	1	+	0	+	0	+	0	+	0	+	0	+	0	= 1

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n+1]$	1		−1		1		1		0		0		0		0		0	
$\langle\vec{r},\vec{s}_1[n+1]\rangle$	0.2	+	−0.2	+	1	+	1	+	0	+	0	+	0	+	0	+	0	$=2$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n]$	1		1		−1		1		1		0		0		0		0	
$\langle\vec{r},\vec{s}_1[n]\rangle$	0.2	+	0.2	+	−1	+	1	+	−1.2	+	0	+	0	+	0	+	0	$=-0.8$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-1]$	0		1		1		−1		1		1		0		0		0	
$\langle\vec{r},\vec{s}_1[n-1]\rangle$	0	+	0.2	+	1	+	−1	+	−1.2	+	1	+	0	+	0	+	0	$=0$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-2]$	0		0		1		1		−1		1		1		0		0	
$\langle\vec{r},\vec{s}_1[n-2]\rangle$	0	+	0	+	1	+	1	+	1.2	+	1	+	1	+	0	+	0	$=\mathbf{5.2}$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-3]$	0		0		0		1		1		−1		1		1		0	
$\langle\vec{r},\vec{s}_1[n-3]\rangle$	0	+	0	+	0	+	1	+	−1.2	+	−1	+	1	+	0.2	+	0	$=0$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-4]$	0		0		0		0		1		1		−1		1		1	
$\langle\vec{r},\vec{s}_1[n-4]\rangle$	0	+	0	+	0	+	0	+	−1.2	+	1	+	−1	+	0.2	+	−0.2	$=-1.2$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-5]$	0		0		0		0		0		1		1		−1		1	
$\langle\vec{r},\vec{s}_1[n-5]\rangle$	0	+	0	+	0	+	0	+	0	+	1	+	1	+	−0.2	+	−0.2	$=1.6$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-6]$	0		0		0		0		0		0		1		1		−1	
$\langle\vec{r},\vec{s}_1[n-6]\rangle$	0	+	0	+	0	+	0	+	0	+	0	+	1	+	0.2	+	0.2	$=1.4$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-7]$	0		0		0		0		0		0		0		1		1	
$\langle\vec{r},\vec{s}_1[n-7]\rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0.2	+	−0.2	$=0$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_1[n-8]$	0		0		0		0		0		0		0		0		1	
$\langle\vec{r},\vec{s}_1[n-8]\rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	−0.2	$=-0.2$

$\text{corr}_{\vec{r}}(\vec{s}_2)[k]$

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+4]$	−1		0		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+4] \rangle$	−0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= −0.2

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+3]$	−1		−1		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+3] \rangle$	−0.2	+	−0.2	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= −0.4

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+2]$	1		−1		−1		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+2] \rangle$	0.2	+	−0.2	+	−1	+	0	+	0	+	0	+	0	+	0	+	0	= −1

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n+1]$	1		1		−1		−1		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+1] \rangle$	0.2	+	0.2	+	−1	+	−1	+	0	+	0	+	0	+	0	+	0	= −1.6

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n]$	1		1		1		−1		−1		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n] \rangle$	0.2	+	0.2	+	1	+	−1	+	1.2	+	0	+	0	+	0	+	0	= 1.6

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-1]$	0		1		1		1		−1		−1		0		0		0	
$\langle \vec{r}, \vec{s}_2[n-1] \rangle$	0	+	0.2	+	1	+	1	+	1.2	+	−1	+	0	+	0	+	0	= 2.4

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-2]$	0		0		1		1		1		−1		−1		0		0	
$\langle \vec{r}, \vec{s}_2[n-2] \rangle$	0	+	0	+	1	+	1	+	−1.2	+	−1	+	−1	+	0	+	0	= −1.2

$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-3]$	0		0		0		1		1		1		−1		−1		0	
$\langle \vec{r}, \vec{s}_2[n-3] \rangle$	0	+	0	+	0	+	1	+	−1.2	+	1	+	−1	+	−0.2	+	0	= −0.4

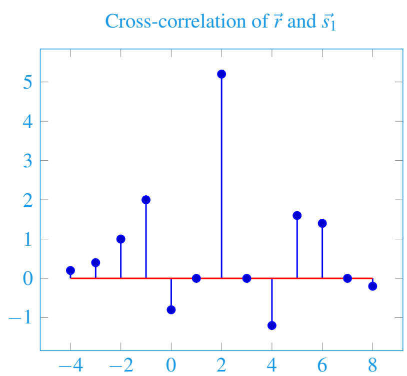
$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-4]$	0		0		0		0		1		1		1		−1		−1	
$\langle \vec{r}, \vec{s}_2[n-4] \rangle$	0	+	0	+	0	+	0	+	−1.2	+	1	+	1	+	−0.2	+	0.2	= 0.8

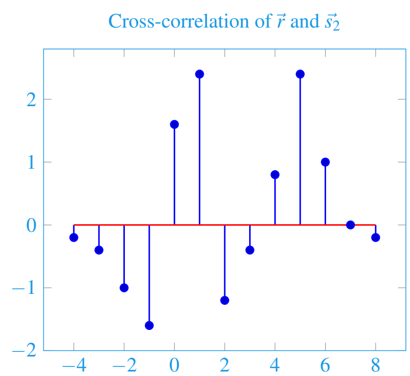
$\vec{r}$	0.2		0.2		1		1		−1.2		1		1		0.2		−0.2	
$\vec{s}_2[n-5]$	0		0		0		0		0		1		1		1		−1	
$\langle \vec{r}, \vec{s}_2[n-5] \rangle$	0	+	0	+	0	+	0	+	0	+	1	+	1	+	0.2	+	0.2	= 2.4

$\vec{r}$	0.2		0.2		1		1		-1.2		1		1		0.2		-0.2	
$\vec{s}_2[n-6]$	0		0		0		0		0		0		1		1		1	
$\langle \vec{r}, \vec{s}_2[n-6] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	1	+	0.2	+	-0.2	$=1$

$\vec{r}$	0.2		0.2		1		1		-1.2		1		1		0.2		-0.2	
$\vec{s}_2[n-7]$	0		0		0		0		0		0		0		1		1	
$\langle \vec{r}, \vec{s}_2[n-7] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0.2	+	-0.2	$=0$

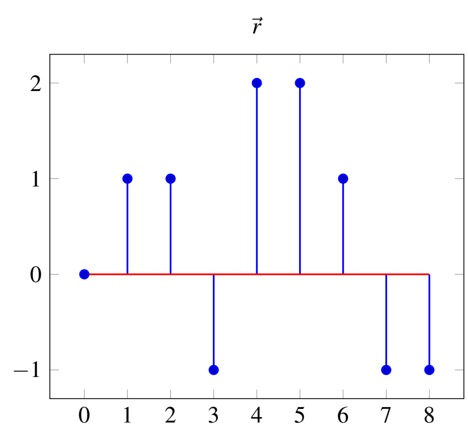
$\vec{r}$	0.2		0.2		1		1		-1.2		1		1		0.2		-0.2	
$\vec{s}_2[n-8]$	0		0		0		0		0		0		0		0		1	
$\langle \vec{r}, \vec{s}_2[n-8] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	-0.2	$=-0.2$





The maximum correlation value is 5.2 at $k = 2$ from satellite 1. Therefore, the transmission likely comes from satellite 1.

(b) Now your cellphone receives a new signal $r[n]$ as below. Can you identify which satellite(s) most likely sent the signal, and by what shift the code is identified relative to our received signal?



Answer: We want to find shifts k_1 and k_2 such that: $\vec{r}[n] = \vec{s}_1[n-k_1] + \vec{s}_2[n-k_2]$.
We calculate both $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ for different shifts k. The index where the maximum correlation value is achieved will tell us the shift indices (delays).

$\text{corr}_{\vec{r}}(\vec{s}_1)[k]$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n+4]$	1	0	0	0	0	0	0	0	0	
$\langle \vec{r}, \vec{s}_1[n+4] \rangle$	0 +	0 +	0 +	0 +	0 +	0 +	0 +	0 +	0	$= 0$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n+3]$	1	1	0	0	0	0	0	0	0	
$\langle \vec{r}, \vec{s}_1[n+3] \rangle$	0 +	1 +	0 +	0 +	0 +	0 +	0 +	0 +	0	$= 1$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n+2]$	-1	1	1	0	0	0	0	0	0	
$\langle \vec{r}, \vec{s}_1[n+2] \rangle$	0 +	1 +	1 +	0 +	0 +	0 +	0 +	0 +	0	$= 2$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n+1]$	1	-1	1	1	0	0	0	0	0	
$\langle \vec{r}, \vec{s}_1[n+1] \rangle$	0 +	-1 +	1 +	-1 +	0 +	0 +	0 +	0 +	0	$= -1$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n]$	1	1	-1	1	1	0	0	0	0	
$\langle \vec{r}, \vec{s}_1[n] \rangle$	0 +	1 +	-1 +	-1 +	2 +	0 +	0 +	0 +	0	$= 1$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n-1]$	0	1	1	-1	1	1	0	0	0	
$\langle \vec{r}, \vec{s}_1[n-1] \rangle$	0 +	1 +	1 +	1 +	2 +	2 +	0 +	0 +	0	$= \mathbf{7}$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n-2]$	0	0	1	1	-1	1	1	0	0	
$\langle \vec{r}, \vec{s}_1[n-2] \rangle$	0 +	0 +	1 +	-1 +	-2 +	2 +	1 +	0 +	0	$= 1$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n-3]$	0	0	0	1	1	-1	1	1	0	
$\langle \vec{r}, \vec{s}_1[n-3] \rangle$	0 +	0 +	0 +	-1 +	2 +	-2 +	1 +	-1 +	0	$= -1$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n-4]$	0	0	0	0	1	1	-1	1	1	
$\langle \vec{r}, \vec{s}_1[n-4] \rangle$	0 +	0 +	0 +	0 +	2 +	2 +	-1 +	-1 +	-1	$= 1$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_1[n-5]$	0	0	0	0	0	1	1	-1	1	
$\langle \vec{r}, \vec{s}_1[n-5] \rangle$	0 +	0 +	0 +	0 +	0 +	2 +	1 +	1 +	-1	$= 3$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-6]$	0		0		0		0		0		0		1		1		-1	
$\langle \vec{r}, \vec{s}_1[n-6] \rangle$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	1	$+$	-1	$+$	1	$=1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-7]$	0		0		0		0		0		0		0		1		1	
$\langle \vec{r}, \vec{s}_1[n-7] \rangle$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	-1	$+$	-1	$=-2$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_1[n-8]$	0		0		0		0		0		0		0		0		1	
$\langle \vec{r}, \vec{s}_1[n-8] \rangle$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	-1	$=-1$

$$\text{corr}_{\vec{r}}(\vec{s}_2)[k]$$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n+4]$	-1		0		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+4] \rangle$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$=0$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n+3]$	-1		-1		0		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+3] \rangle$	0	$+$	-1	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$=-1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n+2]$	1		-1		-1		0		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+2] \rangle$	0	$+$	-1	$+$	-1	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$=-2$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n+1]$	1		1		-1		-1		0		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n+1] \rangle$	0	$+$	1	$+$	-1	$+$	1	$+$	0	$+$	0	$+$	0	$+$	0	$+$	0	$=1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n]$	1		1		1		-1		-1		0		0		0		0	
$\langle \vec{r}, \vec{s}_2[n] \rangle$	0	$+$	1	$+$	1	$+$	1	$+$	-2	$+$	0	$+$	0	$+$	0	$+$	0	$=1$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n-1]$	0		1		1		1		-1		-1		0		0		0	
$\langle \vec{r}, \vec{s}_2[n-1] \rangle$	0	$+$	1	$+$	1	$+$	-1	$+$	-2	$+$	-2	$+$	0	$+$	0	$+$	0	$=-3$

$\vec{r}$	0		1		1		-1		2		2		1		-1		-1	
$\vec{s}_2[n-2]$	0		0		1		1		1		-1		-1		0		0	
$\langle \vec{r}, \vec{s}_2[n-2] \rangle$	0	$+$	0	$+$	1	$+$	-1	$+$	2	$+$	-2	$+$	-1	$+$	0	$+$	0	$=-1$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_2[n-3]$	0	0	0	1	1	1	-1	-1	0	
$\langle \vec{r}, \vec{s}_2[n-3] \rangle$	0	$+$ 0	$+$ 0	$+$ -1	$+$ 2	$+$ 2	$+$ -1	$+$ 1	$+$ 0	$=3$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_2[n-4]$	0	0	0	0	1	1	1	-1	-1	
$\langle \vec{r}, \vec{s}_2[n-4] \rangle$	0	$+$ 0	$+$ 0	$+$ 0	$+$ 2	$+$ 2	$+$ 1	$+$ 1	$+$ 1	$=\mathbf{7}$

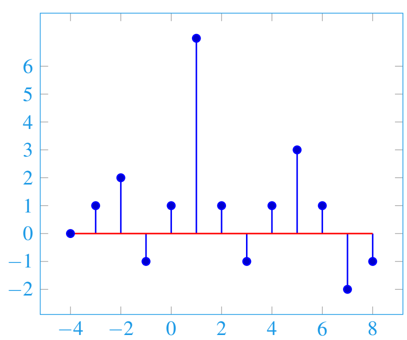
$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_2[n-5]$	0	0	0	0	0	1	1	1	-1	
$\langle \vec{r}, \vec{s}_2[n-5] \rangle$	0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 2	$+$ 1	$+$ -1	$+$ 1	$=3$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_2[n-6]$	0	0	0	0	0	0	1	1	1	
$\langle \vec{r}, \vec{s}_2[n-6] \rangle$	0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 1	$+$ -1	$+$ -1	$=-1$

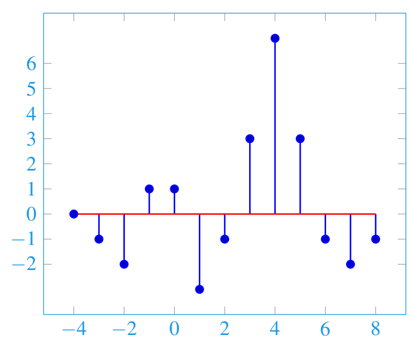
$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_2[n-7]$	0	0	0	0	0	0	0	1	1	
$\langle \vec{r}, \vec{s}_2[n-7] \rangle$	0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ -1	$+$ -1	$=-2$

$\vec{r}$	0	1	1	-1	2	2	1	-1	-1	
$\vec{s}_2[n-8]$	0	0	0	0	0	0	0	0	1	
$\langle \vec{r}, \vec{s}_2[n-8] \rangle$	0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ 0	$+$ -1	$=-1$

Cross-correlation of $\vec{r}$ and $\vec{s}_1$



Cross-correlation of $\vec{r}$ and $\vec{s}_2$



The maximum correlation between signals $\vec{r}$ and $\vec{s}_1$ was achieved at $k_1 = 1$, and the maximum correlation between signals $\vec{r}$ and $\vec{s}_2$ was achieved at $k_2 = 4$.

3. (Practice) Projections Derivation

Let us explore projections in 2D space. Consider two vectors $\vec{a}$ and $\vec{b}$.

Theorem: The point along $\vec{a}$ (in the span of $\vec{a}$) that is closest to $\vec{b}$ is the vector $\vec{z}$ such that $\vec{b} - \vec{z}$ is orthogonal to $\vec{a}$. The vector $\vec{z}$ is given by $\vec{z} = \frac{\langle \vec{a}, \vec{b} \rangle}{||\vec{a}||^2} \vec{a}$.

Hint: Let $\vec{z}$ be the solution to the above theorem. What does it mean for $\vec{z}$ to be in the span of $\vec{a}$?

(a) Derive the theorem using a geometrical interpretation. You may use Figure 1 as the starting point for your proof.

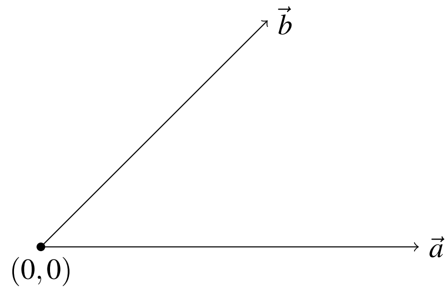
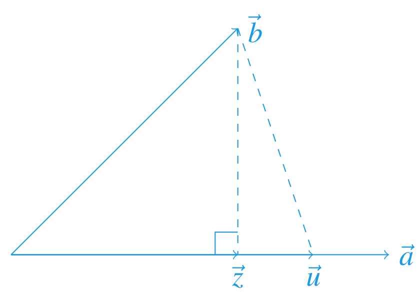


Figure 1: Example vectors $\vec{a}$ and $\vec{b}$ as starting point for geometric proof.

Answer: The point along $\vec{a}$ that is closest to $\vec{b}$ must be orthogonal to $\vec{a}$. We prove this by letting $\vec{z}$ be a vector where $\vec{b}-\vec{z}$ is orthogonal to $\vec{a}$ and also goes through $\vec{b}$. Because $\vec{z}$ is in the span of $\vec{a}$, we may also write $\vec{z}=\beta\vec{a}$, where β is an unknown scalar. Let us choose any other vector $\vec{u}$. We see that $\vec{z}-\vec{b}$, $\vec{u}-\vec{z}$, and $\vec{b}-\vec{u}$ form a triangle. Due to the Pythagorean theorem, the magnitude of $\vec{b}-\vec{u}$ must be greater than the magnitude of $\vec{b}-\vec{z}$. Therefore, the vector $\vec{z}$ is the point along $\vec{a}$ that is closest to $\vec{b}$.

To find the value of β, one approach is to use the Pythagorean theorem. We have:

$$\begin{aligned}
||\vec{z}||^2+||(\vec{b}-\vec{z})||^2 &= ||\vec{b}||^2\\
||\beta\vec{a}||^2+||(\vec{b}-\beta\vec{a})||^2 &= ||\vec{b}||^2\\
||\beta\vec{a}||^2+(\vec{b}-\beta\vec{a})^T(\vec{b}-\beta\vec{a}) &= ||\vec{b}||^2\\
\beta^2||\vec{a}||^2+||\vec{b}||^2-2\beta\langle\vec{a},\vec{b}\rangle+\beta^2||\vec{a}||^2 &= ||\vec{b}||^2\\
2\beta^2||\vec{a}||^2-2\beta\langle\vec{a},\vec{b}\rangle &= 0\\
\beta &= \frac{\langle\vec{a},\vec{b}\rangle}{||\vec{a}||^2}
\end{aligned}$$



(b) Derive the theorem algebraically by solving a problem minimizing the distance between $\vec{b}$ and $\vec{z}$.

Answer: We let $\vec{z} = \beta\vec{a}$. We formulate the minimization problem as:

$$\begin{aligned}
&\min_{\vec{z}=\beta\vec{a}} ||\vec{b} - \vec{z}||^2 \\
=&\min_{\beta} ||\vec{b} - \beta\vec{a}||^2 \\
=&\min_{\beta} (\vec{b} - \beta\vec{a})^T(\vec{b} - \beta\vec{a}) \\
=&\min_{\beta} ||\vec{b}||^2 - 2\beta\langle\vec{a},\vec{b}\rangle + \beta^2||\vec{a}||^2.
\end{aligned}$$

We take the derivative of the above expression with respect to β and set it equal to 0 in order to find the optimal β.

$$\begin{aligned}
2\beta||\vec{a}||^2 - 2\langle\vec{a},\vec{b}\rangle &= 0 \\
\beta &= \frac{\langle\vec{a},\vec{b}\rangle}{||\vec{a}||^2}
\end{aligned}$$