EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 3B

1. Mechanical Inverses

For each sub-part below, determine whether or not the inverse of $\mathbf{A}$ exists.
If it exists, compute the inverse using Gauss-Jordan method.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer: We use Gaussian elimination (also known as the Gauss-Jordan method):

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(b) (PRACTICE)
$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

Answer: We use Gaussian elimination:

$$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_2} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 5 & 4 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 5R_1} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -5 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & -1 & 1 & -5 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 5 \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$.

(c) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Answer: We can again use the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{c}{a}b & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{d-\frac{c}{a}b}R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{c}{a}b} & \frac{1}{d-\frac{c}{a}b} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{b}{a}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{b}{a}\frac{c}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

Therefore, we get that $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
This is a known formula which, if you find useful, you can use for any general 2x2 matrix.
Note that the matrix does not have an inverse if $ad-bc=0$.

(d) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & 4 \end{bmatrix}$

Answer:
Since have a non-square matrix $\mathbf{A}$, there cannot be a unique inverse.
We can understand this from the fact that for $\vec{y} = \mathbf{A}\vec{x}$ the vectors $\vec{x} \in \mathbb{R}^3$ and $\vec{y} \in \mathbb{R}^2$ live in different spaces. This leads us to conclude that there cannot be a unique $\vec{x}$ for each $\vec{y}$.

(e) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Answer: We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right].$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

(f) (PRACTICE)

$$\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$$

Answer:

We use Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow -R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{2}{5} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{5} & 2 & 1 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right].$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{4}{5} & 2 & 1 \\ \frac{2}{5} & -\frac{1}{2} & -1 \\ \frac{1}{5} & -\frac{1}{2} & 0 \end{bmatrix}$.

2. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?
ii. What is the null space of $\mathbf{A}$? What is its dimension?
iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?
iv. Do the columns of $\mathbf{A}$ span $\mathbb{R}^2$? Do they form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer: Column space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.
The column space does not span $\mathbb{R}^2$ and thus are not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$
Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$
Answer:

Column space: $\mathbb{R}^2$

Null space: span $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
The two column spaces are the same as the column span $\mathbb{R}^2$.
This is a basis for $\mathbb{R}^2$.

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$
Answer:

Column space: span $\left\{ \begin{bmatrix} 1 \\ -\frac{3}{2} \end{bmatrix} \right\}$

Null space: span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$
The two column spaces are not the same.
Not a basis for $\mathbb{R}^2$.

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

Answer:

i. The columnspace of the columns is $\mathbb{R}^2$. The columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$. This is because the columns of $\mathbf{A}$ are linearly dependent.

ii. The following algorithm can be used to solve for the null space of a matrix. The procedure is essentially solving the matrix-vector equation $\mathbf{A}\vec{x} = \vec{0}$ by performing Gaussian elimination on $\mathbf{A}$. We start by performing Gaussian elimination on matrix $\mathbf{A}$ to get the matrix into upper-triangular form.

$$\begin{aligned}
\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \text{reduced row echelon form}
\end{aligned}$$

$$\begin{aligned}
&x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0 \\
&x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0 \\
&x_3 \text{ is free and } x_4 \text{ is free}
\end{aligned}$$

Now let $x_3 = s$ and $x_4 = t$. Then we have:

$$\begin{aligned}
x_1 + \frac{1}{2}s - \frac{7}{2}t = 0 \\
x_2 + \frac{5}{2}s + \frac{1}{2}t = 0
\end{aligned}$$

Now writing all the unknowns (x_1, x_2, x_3, x_4) in terms of the dummy variables:

$$\begin{aligned}
x_1 &= -\frac{1}{2}s + \frac{7}{2}t \\
x_2 &= -\frac{5}{2}s - \frac{1}{2}t \\
y &= s \\
z &= t
\end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s \\ -\frac{5}{2}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{2}t \\ -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

So every vector in the nullspace of $\mathbf{A}$ can be written as follows:

$$\text{Nullspace}(\mathbf{A}) = s\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the nullspace of $\mathbf{A}$ is

$$\text{span}\left\{\begin{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\end{bmatrix}\right\}$$

$\mathbf{A}$ has a 2-dimensional null space.

iii. In this case, the column space of the row reduced matrix is also $\mathbb{R}^2$, but this need not be true in general.

iv. No, the columns of $\mathbf{A}$ do not form a basis for $\mathbb{R}^2$.

3. Helpful Guide – Reference Definitions

Vector spaces:

A *vector space* V is a set of elements that is 'closed' under vector addition and scalar multiplication and contains a zero vector. What does closed mean?

That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

More formally, a *vector space* (V, F) is a set of vectors V, a set of scalars F, and two operators that satisfy the following properties:

As a reminder, the mathematical notation $\forall\ \vec{v},\vec{u},\vec{w} \in V$ means *for all possible vectors $\vec{u},\vec{v},\vec{w}$ within the vector space V.*

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w} \quad \forall \quad \vec{v},\vec{u},\vec{w} \in V$.
  - Commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u} \quad \forall \quad \vec{v},\vec{u} \in V$.
  - Additive Identity: There exists an additive identity $\vec{0} \in V$ such that $\vec{v}+\vec{0} = \vec{v} \quad \forall \quad \vec{v} \in V$.
  - Additive Inverse: For any $\vec{v} \in V$, there exists $-\vec{v} \in V$ such that $\vec{v}+(-\vec{v}) = \vec{0}$.
    We call $-\vec{v}$ the additive inverse of $\vec{v}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v} \quad \forall \quad \vec{v} \in V,\ \alpha,\beta \in F$.
  - Multiplicative Identity: There exists $1 \in F$ where $1\cdot\vec{v} = \vec{v} \quad \forall \quad \vec{v} \in F$.
    We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v} \quad \forall \quad \alpha \in F$ and $\vec{u},\vec{v} \in V$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v} \quad \forall \quad \alpha,\beta \in F$ and $\vec{v} \in V$.

Subspaces:

A subset W of a *vector space* V is a *subspace* of V if the above conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace W.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

Basis:

A *basis* for a vector space or subspace is an *ordered set of linearly independent vectors* that *spans the vector space or subspace*.

Therefore, if we want to check whether a set of vectors $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ forms a basis for a vector space V, we check for two important properties:

(a) $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ is linearly independent.

(b) $\operatorname{span}\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\} = V$

As we move along, we'll learn how to identify and construct a basis, and we'll also learn some interesting properties of bases.

Dimension:

The *dimension* of a vector space is the *minimum number* of vectors needed to span the entire vector space. That is, the dimension of a vector space equals the number of vectors in a basis for this vector space.