EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 3A

1. Inverses

In general, the *inverse* of a matrix "undoes" the operation that a matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

Properties of Inverses

For a matrix $\mathbf{A}$, if its inverse exists, then:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1} \quad \text{for a nonzero scalar } k \in \mathbb{R}$$

$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T} \quad T \text{ is "Transpose"}$$

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad \text{assuming } \mathbf{A},\mathbf{B} \text{ are both invertible}$$

(a) Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all invertible matrices.
Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Answer:

Matrix multiplication is *not commutative*, so you cannot flip thwe matrices around within the product. However, it is associative so that you can place parentheses freely:

$$\begin{aligned}
\mathbf{C}^{-1}\,\mathbf{B}^{-1}\,\mathbf{A}^{-1}\;\mathbf{A}\;\mathbf{B}\;\mathbf{C} &= \mathbf{C}^{-1}\,\mathbf{B}^{-1}\left(\mathbf{A}^{-1}\,\mathbf{A}\right)\mathbf{B}\,\mathbf{C} \\
&= \mathbf{C}^{-1}\,\mathbf{B}^{-1}\,\mathbf{I}\,\mathbf{B}\,\mathbf{C} \\
&= \mathbf{C}^{-1}\left(\mathbf{B}^{-1}\,\mathbf{B}\right)\mathbf{C} \\
&= \mathbf{C}^{-1}\,\mathbf{I}\,\mathbf{C} \\
&= \mathbf{C}^{-1}\,\mathbf{C} \\
&= \mathbf{I} \qquad \square
\end{aligned}$$

(b) Now consider the following four matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.
ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.
iii. Are the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ invertible?
iv. Can you find anything in common about the rows (and columns) of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?
(*Bonus:* How does this relate to the invertibility of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?)
v. Are all square matrices invertible?
vi. (PRACTICE) How can you find the inverse of a general $n \times n$ matrix?

Answer:

i.
- $\mathbf{A}$: Preserves the x component and sets the y component to zero.
- $\mathbf{B}$: Preserves the y component and sets the x component to zero.
- $\mathbf{C}$: Replaces the x and y components with the average of the x and y components.
- $\mathbf{D}$: Yields a weighted sum of x and y components. Places the sum in x and twice the sum in y.

ii. Intuitively, none of these operations can be undone because we lost some information. In the first two, we lost one component of the original. In the third case, we replaced both x and y with the average of the two. Thus, different inputs could lead to the same average and we wouldn't be to tell them apart. In the fourth case, we took a weighted sum of the x and y components. There are different values for x and y that could lead to the same sum. However, we cannot recover the original x and y because we didn't compute two unique weighted sums. Instead, we just multiplied the sum by two for the y component of the output.

iii. Since the operations are not one-to-one reversible, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are not invertible.

iv. The rows of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are all linearly dependent. The same is true for the columns. The generalization is that if a matrix is not invertible, then its rows and columns will be linearly dependent.

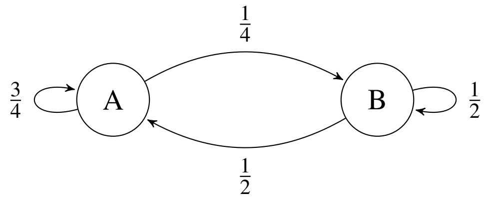
v. No. We have seen in the above parts that there are square matrices that are not invertible.

vi. We know that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. If we treat this as our now familiar $\mathbf{A}\vec{x} = \vec{b}$, we can use Gaussian elimination:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right] \Longrightarrow \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}^{-1} \end{array}\right]$$

2. Transition Matrix

Suppose we have a network of pumps as shown in the diagram below. Let us describe the state of A and B using a state vector $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$ where $x_A[n]$ and $x_B[n]$ are the states at time-step n.



(a) Find the state transition matrix S, such that $\vec{x}[n+1] \;=\; S\,\vec{x}[n]$.
Separately find the sum of the terms for each column vector in S. Do you notice any pattern?

Answer:
We can write the following equations by examining the state transition diagram:

$$\begin{aligned} x_A[n+1] &= (3/4)\,x_A[n] \;+\; (1/2)\,x_B[n] \\ x_B[n+1] &= (1/4)\,x_A[n] \;+\; (1/2)\,x_B[n] \end{aligned}$$

From here we can directly write down the state transition matrix as:

$$S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$$

Note the columns of S each sum to 1, which ensures we have a physical system satisfying conservation.

(b) Let us now find the matrix S^{-1} such that we can recover the previous state $\vec{x}[n-1]$ from $\vec{x}[n]$.
Specifically, solve for S^{-1} such that $\vec{x}[n-1] \;=\; S^{-1}\,\vec{x}[n]$.

Answer: We can use Gauss-Jordan method to solve for matrix S^{-1}:

$$\left[\begin{array}{cc|cc} 3/4 & 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xRightarrow{R_1 \leftarrow \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \xRightarrow{R_2 \leftarrow -4R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right] \xRightarrow{R_2 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right] \xRightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \xRightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \xrightarrow{R_2 \leftarrow -\frac{3}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

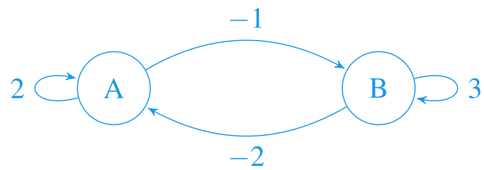
Therefore:

$$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

Note that the columns of S^{-1} still sum to 1 despite matrix elements lying outside $[0,1]$. So while this is not physical, the inverse process obeys conservation as expected.

(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found. Also find the sum of the terms for each column vector in S^{-1}. Do you notice any pattern?

Answer:



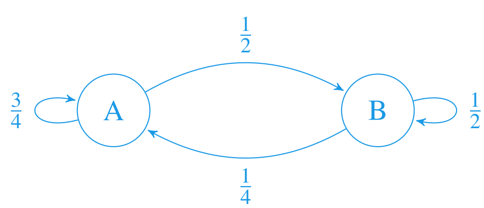
We can write the following equations from the state transition diagram:

$$\begin{aligned} x_A[n-1] &= 2\,x_A[n] - 2x_B[n] \\ x_B[n-1] &= -x_A[n] + 3\,x_B[n] \end{aligned}$$

The matrix S^{-1} can be thought of as the transition that *turns back time* for the pump system. **Although it is non-physical**, the weights that have an absolute value greater than 1 can be thought of as "generating" water, and the weights that have negative weight can be thought of as "destroying" water. However, note (as seen above) that the outflow weights of each node still sum to 1 (i.e. the columns of S^{-1} still sum to 1). This means that in total all of the water is being conserved during the transition between time steps, even when time is reversed.

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?

Answer:



After drawing the "reversed" state transition diagram, we can write the following equations:

$$\begin{aligned} x_A[n+1] &= (3/4)\,x_A[n] + (1/4)\,x_B[n] \\ x_B[n+1] &= (1/2)\,x_A[n] + (1/2)\,x_B[n] \end{aligned}$$

From here, we can directly write down the state transition matrix as:

$$T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

Note that $T \neq S^{-1}$. What we have actually found is that T is equal to the *transpose* of S, denoted by $S^\top$ (the superscript $\top$ denotes the transpose of a matrix). The transpose of a matrix is when its rows become its columns. In general, a matrix's inverse and its transpose are not equal to each other.

(e) Suppose we start in the state $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$. Compute the state vector after 2 time-steps $\vec{x}[3]$.

Answer: There are two way to approach this problem:

i. Compute states successively $\vec{x}[2] = S\,\vec{x}[1]$, then $\vec{x}[3] = S\,\vec{x}[2]$
ii. Compute directly by $\vec{x}[3] = S\,S\,\vec{x}[1]$

They are both equivalent thanks to the fact that matrix multiplication is associative:

$$\vec{x}[3] = S(S\,\vec{x}[1]) = (S\,S)\,\vec{x}[1]$$

We start with method i.

$$\vec{x}[2] = S\,\vec{x}[1] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \cdot 12/4 + 12/2 \\ 1 \cdot 12/4 + 12/2 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$$

$$\vec{x}[3] = S\,\vec{x}[2] = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \cdot 15/4 + 9/2 \\ 1 \cdot 15/4 + 9/2 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix} \qquad \square$$

Alternatively we can use method ii.

$$\vec{x}[3] = S\,S\,\vec{x}[1] \quad \rightarrow \quad S^2 = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 9/16 + 1/8 & 3/8 + 1/4 \\ 3/16 + 1/8 & 1/8 + 1/4 \end{bmatrix} = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix}$$

$$\vec{x}[3] = S^2\,\vec{x}[1] = \begin{bmatrix} 11/16 & 5/8 \\ 5/16 & 3/8 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 11 \cdot 3/4 + 5 \cdot 6/4 \\ 5 \cdot 3/4 + 3 \cdot 6/4 \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix} \qquad \square$$

(f) **(Challenge practice problem)** Given our starting state from the previous problem, what happens if we look at the state of the network after a lot of time steps? Specifically which state are we approaching, as defined below?

$$\vec{x}_{final} = \lim_{n \to \infty} \vec{x}[n]$$

Note that the final state needs to be what we call a *steady state*, meaning $S\,\vec{x}_{final} = \vec{x}_{final}$.
Also what can you say about $x_A[n] + x_B[n]$?
Use information from both of these properties to write out a new system of equations and solve for $\vec{x}_{final}$.

Answer:
We first use the fixed point property to yield an equation relating x_A and x_B for $\vec{x}_{final}$. To see this we subtract $\vec{x}_{final}$ from the fixed point equation to get $S\,\vec{x}_{final} - \vec{x}_{final} = 0$.

To make this easier we recognize that $\vec{x}_{final}$ can be written as the result of the *identity* matrix operation $I\,\vec{x}_{final} = \vec{x}_{final}$.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \rightarrow \quad I\,\vec{x}_{final} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \vec{x}_{final}$$

Thus our expression becomes a system of equations: $(S-I)\vec{x}_{final} = \vec{0}$. Let's see if we can solve it!

$$(S-I)\vec{x}_{final} = \vec{0} \quad \rightarrow \quad \left[\begin{array}{cc|c} 3/4-1 & 1/2-0 & 0 \\ 1/4-0 & 1/2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 1/4 & -1/2 & 0 \end{array}\right]$$

With two row operations ($R_2 \rightarrow R_2 + R_1$ and then $R_1 \rightarrow -4R_1$) we see there are infinite solutions, but at least attain one condition x_A and x_B must satisfy for $\vec{x}_{final}$:

$$\left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 1/4 & -1/2 & 0 \end{array}\right] \xRightarrow{R_2 \rightarrow R_2 + R_1} \left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xRightarrow{R_1 \rightarrow -4R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_A - 2x_B = 0$$

Second, we can look at our computed states from before and notice a pattern:

$$\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix} \rightarrow \vec{x}[2] = \begin{bmatrix} 15 \\ 9 \end{bmatrix} \rightarrow \vec{x}[3] = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix}$$

Most notably $x_A[1] + x_B[1] = 12 + 12 = 24$, $x_A[2] + x_B[2] = 15 + 9 = 24$, and $x_A[3] + x_B[3] = (63 + 33)/4 = 24$. This is no coincidence; it is the result of our conservation property of the network seen by the columns of S summing to 1. So generally $x_A[n] + x_B[n] = 24$ for any n, including the final state.

Now we have two expressions and can solve for $\vec{x}_{final}$:

$$\begin{array}{l} x_A - 2x_B = 0 \\ x_A + x_B = 24 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & 24 \end{array}\right] \xRightarrow{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 3 & 24 \end{array}\right] \rightarrow \begin{array}{c} x_B = 8 \\ x_A = 0 + 2(8) \rightarrow x_A = 16 \end{array}$$

Thus the final state of the system is $\vec{x}_{final} = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$. $\square$