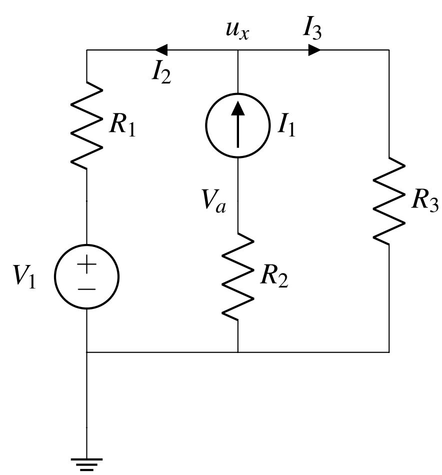
CSM 16A Designing Information Devices and Systems I
Spring 2021 Week 7

1. Superposition

Learning Goal: This problem aims to make students familiar with the technique of superposition. It will also show how to nullify different types of sources in the process.

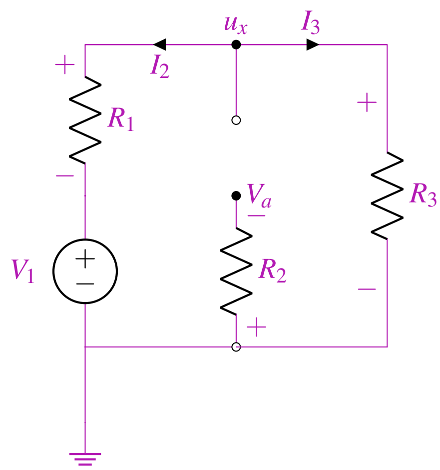
Relevant Notes: **Note 15: Section 15.3** goes over the principle of superposition. Some intuition behind why superposition works is that we can calculate the result from each source independently and add the results up (Note 15). Think of the multiple sources (ex. voltage source, current source, etc.) like basis vectors that are orthogonal to each other or equations that are linearly independent - in other words they have no relation to each other. So we can add them up to get our final result!

Solve the following circuit for u_x using superposition. Let $R_1 = 10\Omega$, $R_2 = 5\Omega$, $R_3 = 2\Omega$, $V_1 = 12V$, and $I_1 = 3A$.



(a) Find u_x when only V_1 is active.

Answer: We start off our analysis using superposition by nullifying all independent sources except for one. In this part, we nullify the current source I_1, replacing it with a open circuit. We know its an open circuit because the I vs V graph of a current source is a horizontal line with the value of current supplied by the source. Zeroing this out shifts the line to along the V axis (basically no current, any voltage value), which is like an open circuit:



Now, all the current flows through I_2 and I_3, with nothing going through the open circuit or R_2. Our circuit has been reduced to a single loop, with elements V_1, R_1, and R_3 in series. Notice that this is a voltage divider! Thus, we can write

$$V_{R_3} = V_1 \frac{R_3}{R_1 + R_3}$$

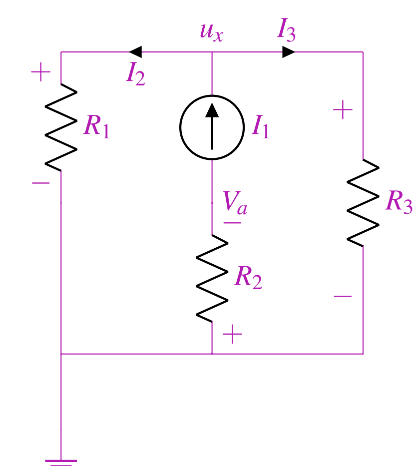
Plugging in our numerical values gives us

$$V_{R_3} = (12\text{V}) \frac{(2\Omega)}{(10\Omega) + (2\Omega)} = 2\text{V}$$

Now $V_{R_3} = u_x - 0 = u_x$, so $u_x = 2\text{V}$

(b) Find u_x when only I_1 is active.

Answer: For this part, we continue our analysis using superposition to find u_x by nullifying the voltage source, which puts a short circuit in its place. We know its an short circuit because the I vs V graph of a voltage source is a vertical line with the value of voltage supplied by the source. Zeroing this out shifts the line to along the I axis (basically no voltage, any current value), which is like a short circuit:



First, we do KCL on the node at u_x:

$$I_1 = I_2 + I_3$$

Next, we use Ohm's Law on the resistor R_1. We know one end of the resistor is at voltage u_x and the other end is connected to ground, so:

$$V = IR$$

$$u_x - 0 = I_2 R_1$$

$$I_2 = \frac{u_x}{R_1}$$

Likewise, we can use Ohm's Law on the resistor R_3:

$$V = IR$$

$$u_x - 0 = I_3 R_3$$

$$I_3 = \frac{u_x}{R_3}$$

We can substitute both I_2 and I_3 into the KCL equation to solve for u_x:

$$I_1 = \frac{u_x}{R_1} + \frac{u_x}{R_3}$$

$$I_1 = u_x(\frac{1}{R_1} + \frac{1}{R_3})$$

$$u_x = I_1 \frac{R_1 R_3}{R_1 + R_3}$$

Now, we plug in numerical values:

$$u_x = (3\text{A}) \frac{(10\Omega)(2\Omega)}{(10\Omega) + (2\Omega)}$$

$$= 5\text{V}$$

(c) Use your results from the last two parts to find u_x when all the sources are active.

Answer: We have found each individual component using superposition, by zeroing out the current source in part (a) and the voltage source in part (b). Now, to find u_x when all sources are active, we add together the u_x's we found in previous parts. We will also show the algebra here:

$$u_x = u_{x,a} + u_{x,b}$$

$$= V_1 \frac{R_3}{R_1 + R_3} + I_1 \frac{R_1 R_3}{R_1 + R_3}$$

$$= \frac{R_3(V_1 + I_1 R_1)}{R_1 + R_3}$$

From here, we can find u_x either by plugging in values for V_1, I_1, R_1, and R_3, or taking the answers from part (a) and part (b) and adding them:

$$u_x = 2\text{V} + 5\text{V} = 7\text{V}$$

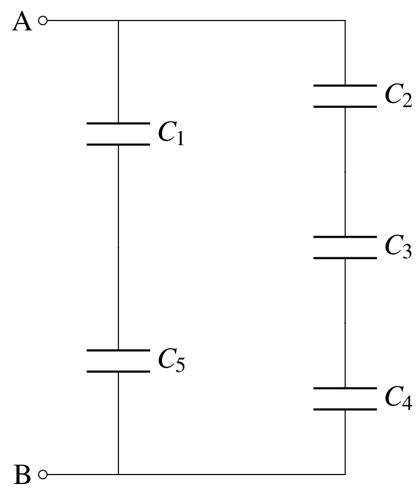
2. Equivalence in Capacitive Networks

Learning Goal: This objective of this problem is to practice finding equivalent capacitance for series/parallel network of capacitors.

Relevant Notes: **Note 16** derives the equivalent capacitance formula for series/ parallel capacitors.

For all of the following networks find an expression or a numerical value for the equivalent capacitance between terminals A and B.

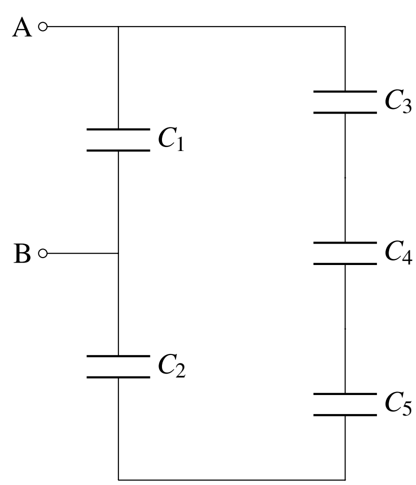
(a)



Answer: Here we have two branches connected in parallel, one including capacitors C_1, C_5 (which are connected in series) and one including capacitors C_2, C_3, and C_4 (which are also connected in series). The equivalent capacitance of the left branch is $C_1||C_5$, where $||$ is the parallel operator (i.e. $a||b = \frac{ab}{a+b}$). Similarly, for the right branch, the equivalent capacitance is $C_2||C_3||C_4$. Since the two branches are in parallel, we can sum up their equivalent capacitances:

$$C_{AB} = (C_1||C_5) + (C_2||C_3||C_4)$$

(b)



Answer:

Here, we have two branches connected in parallel, with the first branch containing only C_1. The second branch contains C_3, C_4, C_5, and C_2 in series. The equivalent capacitance of the right branch is $C_2||C_3||C_4||C_5$. Then, we can sum this up with C_1 from the left branch to get:

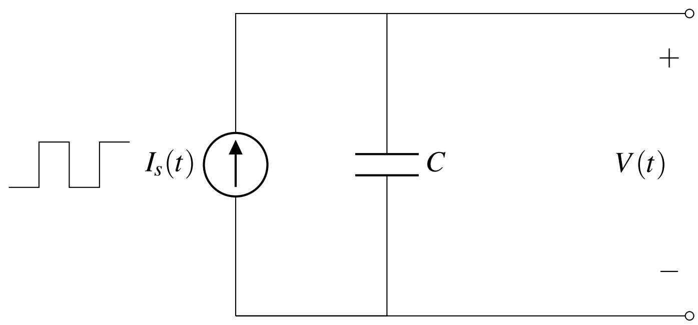
$$C_{AB} = C_1 + (C_2||C_3||C_4||C_5)$$

3. Capacitor with a Periodic Current Source

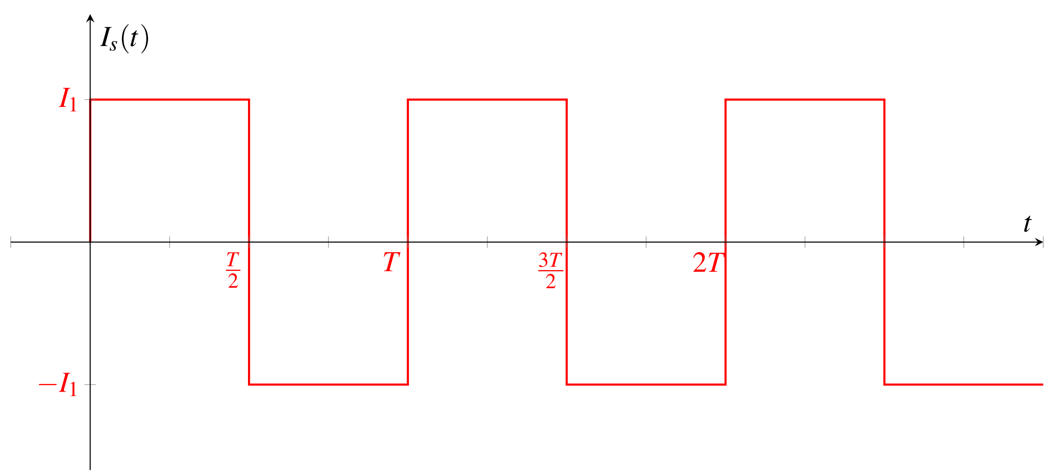
Learning Goal: This problem aims to make students familiar with the charging/ discharging response of a capacitor.

Relevant Notes: Note 17 covers capacitive behavior in the presence of difference types of current sources.

Capacitive touchscreen requires detection of capacitance change due to touch. If we connect a known current source I_s to the capacitor and measure the voltage across the capacitor V, we will be able to solve for the capacitance C. So we build the following circuit to measure with a periodic current source:



(a) Let us assume the current I_s is a function of time as follows:



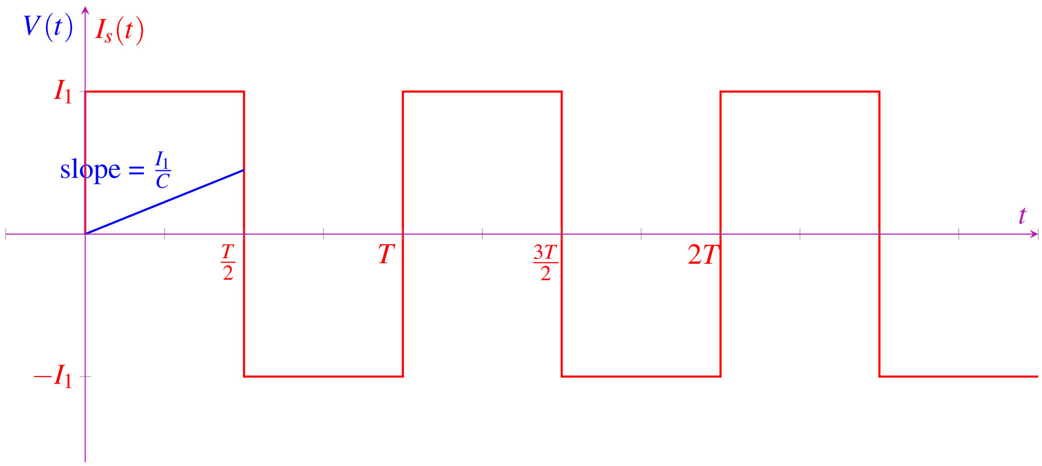
What does the voltage V look like with this current source? Let's assume that the capacitor is initially uncharged (i.e. $Q = 0$). Since $Q = CV$, this means that at time $t = 0$ the voltage $V = 0$.

Answer: When a constant current source is applied to a capacitor, we know that the voltage obeys the following equation

$$V_C(t) = \frac{I}{C}t + V_C(0). \tag{1}$$

Our periodic current source I_s is constant from $t = 0$ to $t = \frac{T}{2}$, so we can apply equation 1 over this time period. We know the initial voltage is zero, so:

$$V(t) = \frac{I_1}{C}t \quad \text{when } 0 \leq t \leq \frac{T}{2}$$



In order to figure out what happens next, let's consider a more generic version of equation 1:

$$V_C(t) = \frac{I}{C}(t - t_0) + V_C(t_0). \tag{2}$$

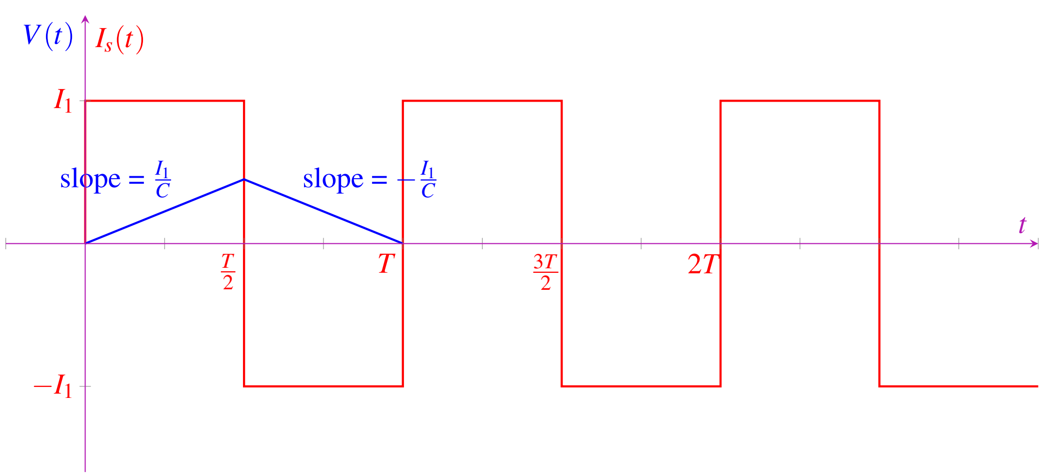
With this equation, we can consider an arbitrary starting time t_0 instead of always starting at $t = 0$. Plugging in $t_0 = 0$ yields equation 1. Like equation 1, the above equation is only true when the current is constant.

The next time period with constant current is from $t = \frac{T}{2}$ to $t = T$. Over this time, the current through the capacitor is $-I_1$. Since we are starting at time $\frac{T}{2}$, we set $t_0 = \frac{T}{2}$ and plug into equation 2.

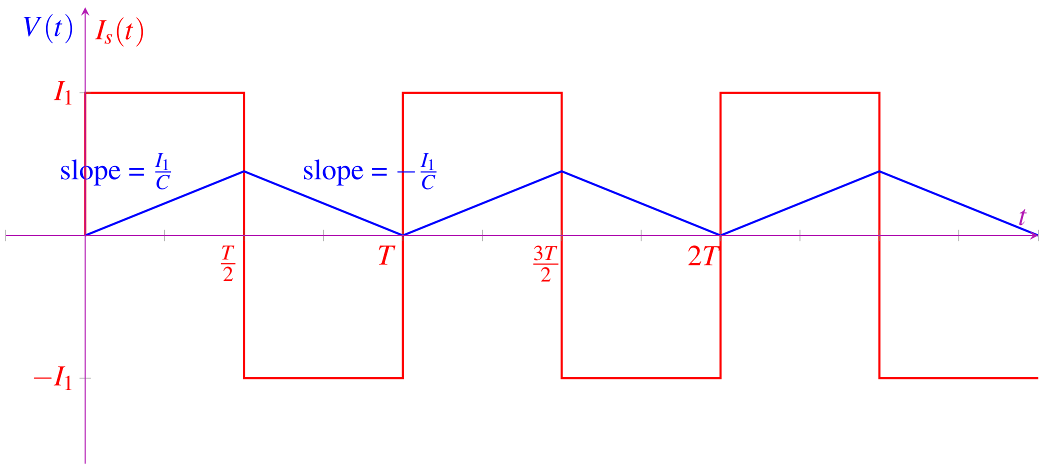
$$V(t) = \frac{-I_1}{C}\left(t - \frac{T}{2}\right) + V\left(\frac{T}{2}\right)$$
$$V(t) = \frac{-I_1}{C}\left(t - \frac{T}{2}\right) + \frac{I_1 T}{2C}$$

Combining with our previous relationship yields:

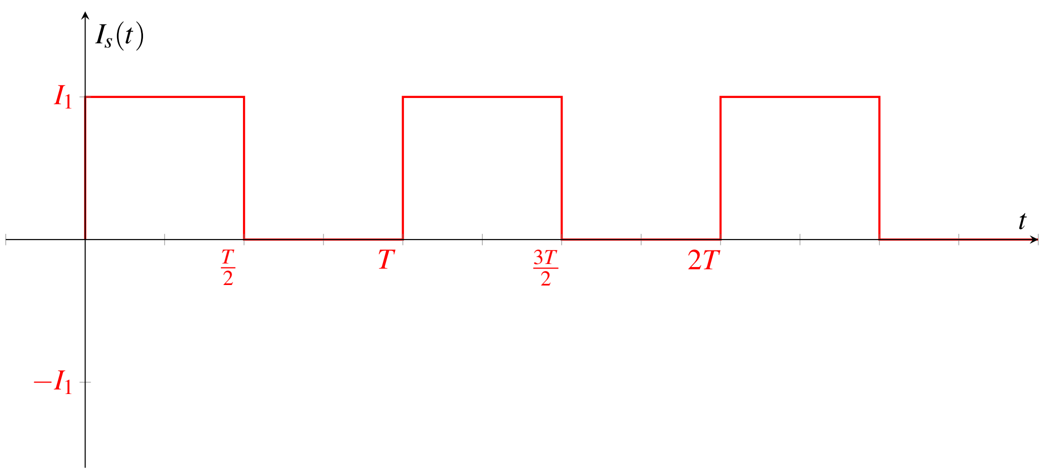
$$V_C(t) = \begin{cases} \frac{I_1}{C}t & \text{when } 0 \leq t \leq \frac{T}{2} \\ \frac{-I_1}{C}\left(t - \frac{T}{2}\right) + \frac{I_1 T}{2C} & \text{when } \frac{T}{2} < t \leq T \end{cases}$$



To determine the full behavior of $V(t)$, we could continue to apply equation 2 for each period of constant current. However, we notice that at $t = T$, the voltage and current are the same as they were at $t = 0$. Since the current source is periodic (repeats every T), the voltage pattern will also repeat.



(b) Now let us assume the current I_s is a function of time as follows:



What does the voltage V qualitatively look like with this current source? Draw out on the above graph how the voltage changes over time, starting at time $t = 0$. Let's assume that the capacitor is initially uncharged (i.e. $Q = 0$). Since $Q = CV$, this means that at time $t = 0$ the voltage $V = 0$.

Answer:

In the first segment, when $0 \leq t \leq \frac{T}{2}$, we found in part (a) using equation 2 that

$$V_C(t) = \frac{I}{C}t$$

However, when $\frac{T}{2} < t \leq T$, $I_s(t)$ is now 0. We again use equation 2 from part (a)

$$V_C(t) = \frac{I}{C}(t - t_0) + V_C(t_0).$$

knowing that $t_0 = \frac{T}{2}$. To find $V_C(t_0)$, we can use the fact that $0 \leq t \leq \frac{T}{2}$ when $t = t_0$ to get $V_c(t_0) = \frac{I_1}{C}\frac{T}{2}$. Plugging these values into equation 2 gives:

$$V_C(t) = \frac{0}{C}(t - \frac{T}{2}) + \frac{I_1}{C}\frac{T}{2} = \frac{I_1 T}{2C}$$

Qualitatively, this indicates that $V_C(t)$ is held constant whenever $I_s(t) = 0$. However, when $I_s(t) = I_1$, $V_C(t)$ goes upwards with slope $\frac{I_1}{C}$, starting from the constant value of $V_C(t)$ while its slope is 0.