CSM 16A Designing Information Devices and Systems I
Spring 2021 Week 4

1. **Eigen Introduction**

Learning Goal: The goal of this problem is to practice both intuitively and mechanically finding eigenvalues and their corresponding eigenvectors/eigenspaces.

Relevant Notes: **Note 9 Sections 9.2, 9.4, and 9.6** cover the process of finding eigenvalue-eigenvector pairs.

(a) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(b) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

(c) Consider a matrix that rotates a vector in $\mathbb{R}^2$ by 45° counterclockwise about the origin in a coordinate plane. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. This matrix is given as

$$\mathbf{E} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

What are the eigenvalues and eigenvectors of this matrix?

(d) Solve for the eigenvalue-eigenvector pairs for the following 2 by 2 matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Also find the eigenspaces.

(e) Find the eigenvectors for matrix $\mathbf{A}$ given that we know that $\lambda_1 = 4, \lambda_2 = \lambda_3 = -2$ and that

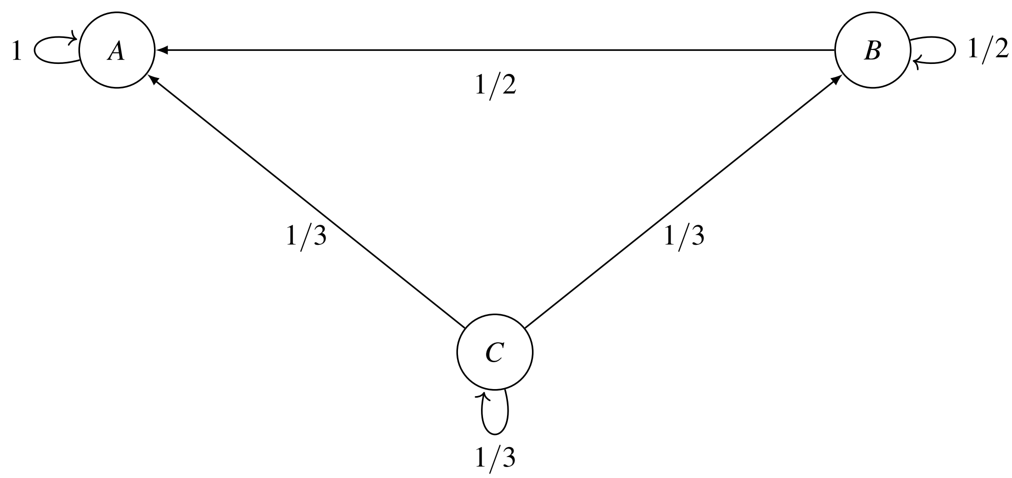
$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Also find the eigenspaces.

2. Page Rank

Learning Goal: This problem is designed to provide insight into state transition. We will observe how the steady state depends of the eigenvalue and eigenvectors of a state-transition matrix.

Now suppose we have a network consisting of 3 websites connected as shown below. Each of the weights on the edges represent the probability of a user taking that edge.



(a) Call the transition matrix for this system **P** Write down **P** from this graph. (*Hint: Try to recall the properties of transition matrices and observe the sum of each column*).

(b) We want to rank these webpages in order of importance. Can you predict at least one of the eigenvalues of **P**? Verify your predicted eigenvalue by calculation and then find the corresponding eigenvectors of **P**.

(c) Now you are told that the other two eigenvalues of $\mathbf{P}$ are $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = \frac{1}{3}$, and the corresponding eigenvectors are $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, respectively.
Suppose we start with just 30 users in A and no users in B and C. Can you express the initial state, $\vec{x}[0]$, as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$?

(d) Now use the results from the previous part to express the state at time step n as a function of the eigenvectors and eigenvalues. What is the steady-state? Is the steady-state different from the initial state? Why?
Relevant Notes: **Note 9: Subsection 9.8.2** are helpful for this problem.

(e) Now suppose we start with 30 users in A, 30 users in B and no users in C. Express the initial state, $\vec{x}[0]$, as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ and find the steady state. Is the steady-state different from the initial state? Why?

(f) Suppose that we start with 90 users evenly distributed among the websites. Without doing any calculations, can you estimate the steady-state number of people who will end up at each website?

3. Diagonalization and Change of Basis

Learning Goal: The goal of this problem is to understand how to perform change of basis and diagonalization computations. Please look into **Note 10** for more on Diagonalization and Change of Basis.

(a) Let $A = \mathbb{R}^2$, **B** = $\left\{ \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$, **C** = $\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$, and $\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

i. Find $[x]_B$.

ii. Find $P_{C \leftarrow B}$.

iii. Compute $[x]_C$, given that you only know $[x]_B$ and $P_{C\leftarrow B}$.

iv. What is the relationship between matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$. Prove it.

(b) Let matrix $\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

i. Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $A = PDP^{-1}$

ii. What is $\mathbf{D}^{2021}$?

