CSM 16A Designing Information Devices and Systems I
Spring 2021 Week 3

1. A Tale of Two Spaces

Learning Goal: The goal of this problem is to understand subspaces, basis vectors, and dimension. Please look into **Note 8 Section 8.1** for more on subspaces.

(a) Consider the set U, which is a subset of $\mathbb{R}^3$, defined below. Is U a subspace?

$$U = \left\{ \begin{bmatrix} x \\ 0 \\ x+y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$$

Answer: In order to check whether U is a subspace or not, we need to check three properties: the set contains the zero vector, is closed under vector addition, and closed under scalar multiplication. First, U **must contain the zero vector of** $\mathbb{R}^3$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Essentially, we want to find some $x, y \in \mathbb{R}$ such that

$$\begin{bmatrix} x \\ 0 \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system, we get $x = 0$ and $y = 0$. The existence of a solution for (x, y) indicates that the zero vector is in the set U.

Next, we must check for **closure under vector addition**, which means given any vectors $\vec{u}_1, \vec{u}_2 \in U$, then it must be true that $\vec{u}_1 + \vec{u}_2 \in U$. Since $\vec{u}_1$ and $\vec{u}_2$ are in U, we can write them as

$$\vec{u}_1 = \begin{bmatrix} x_1 \\ 0 \\ x_1 + y_1 \end{bmatrix}$$

$$\vec{u}_2 = \begin{bmatrix} x_2 \\ 0 \\ x_2 + y_2 \end{bmatrix}$$

Then,

$$\vec{u}_1 + \vec{u}_2 = \begin{bmatrix} x_1 + x_2 \\ 0 \\ x_1 + y_1 + x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \\ (x_1 + x_2) + (y_1 + y_2) \end{bmatrix}$$

This resulting vector is in the form $\begin{bmatrix} x \\ 0 \\ x+y \end{bmatrix}$ where $x = x_1 + x_2$ and $y = y_1 + y_2$. Therefore, it must also be in U, so we have confirmed that U is closed under vector addition.

Finally, we must check for **closure under scalar multiplication**, which means given any vector $\vec{u} \in U$ and any scalar $\alpha \in \mathbb{R}$, then it must be true that $\alpha\vec{u} \in U$. We can write $\vec{u}$ in the form

$$\vec{u} = \begin{bmatrix} x_1 \\ 0 \\ x_1 + y_1 \end{bmatrix}$$

. Then,

$$\alpha\vec{u} = \begin{bmatrix} \alpha x_1 \\ 0 \\ \alpha(x_1 + y_1) \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ 0 \\ \alpha x_1 + \alpha y_1 \end{bmatrix}$$

This resulting vector is in the form $\begin{bmatrix} x \\ 0 \\ x+y \end{bmatrix}$ where $x = \alpha x_1$ and $y = \alpha y_1$. Therefore, it must also be in U, so we have confirmed that U is closed under scalar multiplication. Since U satisfies all three properties, it is a subspace.

(b) Find a basis for U. What is its dimension?

Answer: We can write any vector in U as

$$\begin{bmatrix} x \\ 0 \\ x+y \end{bmatrix}$$

for some $x, y \in \mathbb{R}$. This can then be written as

$$U = \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

. So a set of basis vectors is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

. The dimension is then the number of basis vectors, 2.

(c) Consider the set V, which is a subset of $\mathbb{R}^3$, defined below. Is V a subspace?

$$V = \left\{ \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix} \mid z \in \mathbb{R} \right\}$$

Answer: Yes, V is a subspace. We can confirm this by following the steps from part (a):
Existence of zero vector: When $z = 0$, the form in the definition of V becomes the zero vector in $\mathbb{R}^3$.
Closure under addition: Given arbitrary vectors $\vec{v_1}, \vec{v_2} \in V$, we can write them as

$$\vec{v_1} = \begin{bmatrix} 0 \\ z_1 \\ 0 \end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix} 0 \\ z_2 \\ 0 \end{bmatrix}$$

Then,

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 0 \\ z_1 + z_2 \\ 0 \end{bmatrix}$$

which is in V. So V is closed under addition.
Closure under scalar multiplication: Given arbitrary vector $\vec{v}_1 \in V$ and scalar $\alpha \in \mathbb{R}$, we can write

$$\alpha \vec{v}_1 = \begin{bmatrix} 0 \\ \alpha z_1 \\ 0 \end{bmatrix}$$

which is in V. So V is closed under scalar multiplication. Since V satisfies all three properties, it is a subspace.

(d) Find a basis for V. What is its dimension?

Answer: We can write any vector in V as

$$\begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix}$$

for some $z \in \mathbb{R}$. This can then be written as

$$V = \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix} = z \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

. So a set of basis vectors is:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

. The dimension is the number of basis vectors, 1.

(e) Can you express the basis vector(s) you found in part (d) as a linear combination of the basis vector(s) you found in part (b)? Why or why not?

Answer: Let the basis vectors in part (b) be $\vec{b}_1, \vec{b}_2$ and the basis vector in part (d) be $\vec{b}_3$. To see if we can express $\vec{b}_3$ as a linear combination of $\vec{b}_1, \vec{b}_2$, we seek scalars $m, n \in \mathbb{R}$ such that $m\vec{b}_1 + n\vec{b}_2 = \vec{b}_3$, or:

$$m \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} m \\ 0 \\ m+n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Focusing on the second row, we have 0 on the left hand side of the equation and 1 on the right hand side. This is not possible, so the basis vectors found in part (b) are linearly independent to the basis vector in part (d). The only vector that can be described by both sets of basis vectors is the zero vector.

Something further to think about: can you express any arbitrary vector in V in terms of the basis vectors of U? Or vice versa?

2. Nullspace and Loss of Dimensionality [WALK-THROUGH]

Learning Goal: The goal of this problem to understand the relationship between nullspace and loss of dimensionality/ invertibilty.

Please look into **Note 8 Section 8.3** to learn how the dimension of the output space depends on the nullspace.

Answer the following questions for all three parts:

- Find the columnspace and nullspace of the following matrices in terms of basis vectors.
- What are the dimensions of the columnspace/nullspace? Remember that the Rank Nullity theorem shows that the number of columns of a matrix A = dim(N(A)) [nullity of matrix A] + dim(C(A)) [rank of matrix A]
- What kind of geometry is represented by the columnspace/nullspace?
- Is the matrix invertible?

(a) Consider a matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer: We see that the matrix $\mathbf{P}$ is already in the row-reduced form. We can see that all the columns are linearly independent. So they can form a basis for the columnspace. The basis for columnspace is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So the columnspace can be written as:

$$\mathrm{C}(\mathbf{P}) = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since there are three basis vector for the columspace, the dimension of columnspace is 3, i.e.

$$\dim(\mathrm{C}(\mathbf{P})) = 3$$

The columnspace represents a 3 dimensional volume, i.e. $\mathbb{R}^3$.
In order to find the nullspace, we can use the augmented matrix to solve for $\mathbf{P}\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So the solution is $\vec{x} = \vec{0}$. The nullspace can be written as:

$$\mathrm{N}(\mathbf{P}) = \{\vec{0}\}$$

Since the nullspace is the zero vector (which is considered dimensionless), the dimension of nullspace is 0, i.e.

$$\dim(\mathrm{N}(\mathbf{P})) = 0$$

We call this nullspace **trivial**. The nullspace represents a point in $\mathbb{R}^3$, which is dimensionless.
Since the columns are linearly independent, the matrix is invertible.

(b) Consider a matrix $\mathbf{Q}$:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Answer: We see that the matrix $\mathbf{Q}$ is already in the row-reduced form. We can see that the first two columns are linearly independent. The third column is the summation of first and second column. So the first and second columns can form a basis for the columnspace. The basis for columnspace is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

So the columnspace can be written as:

$$\mathrm{C}(\mathbf{Q}) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since there are two basis vector for the columspace, the dimension of columnspace is 2, i.e.

$$\dim(\mathrm{C}(\mathbf{Q})) = 2$$

The columnspace represents a 2 dimensional plane inside $\mathbb{R}^3$.
In order to find the nullspace, we can use the augmented matrix to solve for $\mathbf{Q}\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If we consider x_3 the free variable, we can evaluate x_1 and x_2:

$$x_1 = -x_3$$
$$x_2 = -x_3$$

So the solution is $\vec{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. The nullspace can be written as:

$$\mathrm{N}(\mathbf{Q}) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Since nullspace has one basis vector, the dimension of nullspace is 1, i.e.

$$\dim(\mathrm{N}(\mathbf{Q})) = 1$$

The nullspace represents a straight line in $\mathbb{R}^3$.
Since the columns are not linearly independent, the matrix is not invertible.

(c) Consider a matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Answer: We see that the matrix $\mathbf{Q}$ is already in the row-reduced form. We can see that both the second and third columns are the scalar multiples of the first column. So only one column is linearly independent (we can choose either, here let's choose the first column). The basis for columnspace is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

So the columnspace can be written as:

$$\mathrm{C}(\mathbf{M}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Since there is one basis vector for the columspace, the dimension of columnspace is 1, i.e.

$$\dim(\mathrm{C}(\mathbf{M})) = 1$$

The columnspace represents a 1-dimensional line inside $\mathbb{R}^3$.
In order to find the nullspace, we can use the augmented matrix to solve for $\mathbf{M}\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If we consider both x_2 and x_3 to be the free variables, we can evaluate x_1:

$$x_1 = -2x_2 - 3x_3$$

So the solution is $\vec{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. The nullspace can be written as:

$$\mathrm{N}(\mathbf{M}) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since nullspace has two basis vectors, the dimension of nullspace is 2, i.e.

$$\dim(\mathrm{N}(\mathbf{M})) = 2$$

The nullspace represents a plane in $\mathbb{R}^3$.
Since the columns are not linearly independent, the matrix is not invertible.

3. Fundamental Subspaces [WALK-THROUGH]

Learning Goal: The goal of this problem to practice finding the columnspace and nullspace of a matrix. Please look into **Note 8 Section 8.2-8.4** to learn about the significance of columnspace and nullspace.

Consider a matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -2 & -1 \end{bmatrix}$$

(a) Find a basis for the column space of $\mathbf{A}$. What is the dimension of this space?

Answer:

We row reduce $\mathbf{A}$:

$$\begin{aligned} \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -2 & -1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \text{ using } R_3 \leftarrow R_3 + R_1 \\ &\rightarrow \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ using } R_3 \leftarrow R_2 + R_3 \\ &\rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ using } R_1 \leftarrow R_1 + 2R_2 \end{aligned}$$

We see that our pivots are in the first and second columns, so we can construct the column space using the first and second columns of A.

$$\mathrm{C}(\mathbf{A}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The dimension of this space is 2.

(b) Find a basis for the nullspace of $\mathbf{A}$. What is the dimension of this space?

Answer:

Recall that the nullspace is set of all $\vec{x}$ such that $\mathbf{A}\vec{x} = \vec{0}$. Let the entries of $\vec{x}$ be x_1, x_2, x_3, x_4.

Use the row reduced echelon form of $\mathbf{A}$ from the previous part of the problem, augmented with $\vec{0}$:

$$\left[\begin{array}{c|c} \mathbf{A} & \vec{0} \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that the third and fourth column do not have pivots, i.e. x_3 and x_4 are the free variables. We can write the equations that this matrix represents as follows:

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 = x_3 = 0 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

$$\begin{cases} x_1 = -x_3 - x_4 \\ x_2 = x_3 \\ 0 = 0 \end{cases}$$

Rewrite $\vec{x}$:

$$\begin{aligned} \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} -x_3 - x_4 \\ x_3 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} \\ &= x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

We can construct our nullspace from the above vectors:

$$\mathrm{N}(\mathbf{A}) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dimension of this space is 2.

4. Proof on Nullspace

Learning Goal: The goal of this problem is to practice some more proof development skills.

(a) **Show that if a square matrix A is invertible, then it has a trivial nullspace.**

Please look into **Note 8 Section 8.3** to learn how the dimension of the output space depends on the nullspace.

Answer:

We know that the square matrix $\mathbf{A}$ is invertible. We want to show that the square matrix $\mathbf{A}$ has a trivial nullspace.

Steps to get there:

A trivial nullspace means that the vector solution to the equation, $\mathbf{A}\vec{x} = \vec{0}$, is the $\vec{0}$ vector.

Assume that $\mathbf{A}$ is invertible and let $\vec{x}$ be a vector, in $\mathrm{N}(\mathbf{A})$, such that $\mathbf{A}\vec{x} = \vec{0}$. Multiplying both sides of the equation by the inverse of $\mathbf{A}$, will yield:

$$(\mathbf{A}^{-1})\mathbf{A}\vec{x} = (\mathbf{A}^{-1})\vec{0}$$

The inverse of a matrix multiplied by the matrix itself is just the identity matrix; so, we are left with:

$$\mathbf{I}\vec{x} = \vec{0}$$
$$\vec{x} = \vec{0}$$

Since $\vec{x} = \vec{0}$ is the only solution to this equation, we've shown that if a square matrix is invertible, then it has a trivial nullspace.

5. Non-invertible Square Matrix

Learning Goal: The goal of this problem is to understand loss of dimensionality in relation to nullspace.

(a) For given matrices $\mathbf{A} \in \mathbb{R}^{3\times 2}$ and $\mathbf{B} \in \mathbb{R}^{2\times 3}$, the products will be square matrices: $\mathbf{AB} \in \mathbb{R}^{3\times 3}$ and $\mathbf{BA} \in \mathbb{R}^{2\times 2}$. Show that $\mathbf{AB}$ is not invertible.

Please look into **Note 8 Section 8.3** to learn how the dimension of the output space depends on the nullspace.

Hint: *A good proof strategy is to utilize what we have already proven before. Is there a way we can use the result in Question 4, "Proof on Nullspace"?*

Answer: We know that $\mathbf{A} \in \mathbb{R}^{3\times 2}$ and $\mathbf{B} \in \mathbb{R}^{2\times 3}$. We want to show that $\mathbf{AB}$ is not invertible.

Steps to get there: There are many ways to do this proof, and we will only show one possible method here, a proof by contradiction.

Assume, for sake of contradiction, that $\mathbf{AB}$ is invertible. Since $\mathbf{AB} \in \mathbb{R}^{3\times 3}$, it is square, so by the theorem proved in Problem 4, $\mathbf{AB}$ must have a trivial nullspace. This means that $\vec{x} = \vec{0}$ is the only solution to the equation

$$\mathbf{AB}\vec{x} = \vec{0}$$

Now, we are also given that $\mathbf{B} \in \mathbb{R}^{2\times 3}$. $\mathbf{B}$ has 3 column vectors, each of which are $\in \mathbb{R}^2$. The transformation $\mathbf{B}$ results in a loss of dimensionality: the dimension of $\mathbb{R}^2$ is 2, so at maximum, two of the column vectors will span $\mathbb{R}^2$. This means that we are guaranteed that at least one of the column vectors is linearly dependent, so $\mathbf{B}$ has a nontrivial nullspace. By definition of nontrivial nullspace, there exists $\vec{x} \neq \vec{0}$ that solves the equation $\mathbf{B}\vec{x} = \vec{0}$. Call this solution x':

$$\mathbf{B}\vec{x'} = \vec{0}$$

Left-multiplying both sides of the equation by $\mathbf{A}$ gives us

$$\mathbf{AB}\vec{x'} = \vec{0}$$

But this contradicts the result we achieved earlier in the proof: then, we found that the only solution to the equation $\mathbf{AB}\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, and now, we have shown there must exist a nonzero solution, x'! These two statements are contradictory; therefore, there must be something wrong with our initial assumption that $\mathbf{AB}$ is invertible. It follows that $\mathbf{AB}$ is non-invertible.