CSM 16A Designing Information Devices and Systems I
Spring 2021 Week 2

1. **Proof on Linear (In)Dependence** [WALK-THROUGH]

   Learning Goal: The goal of this problem is to practice some proof development skills.

   (a) **Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{0}$, has a non-zero solution, then the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ are linearly dependent.**

   We are going to use the approach outlined in **Note 4**. Please also look into **Note 3 Subsection 3.1.1** for the definition of linear dependence/ independence.

   (i) **Start with what we already know:**
   We know that system of equations, $\mathbf{A}\vec{x} = \vec{0}$, has a non-zero solution, $\vec{u}$. Express this information in a mathematical form.

   (ii) **Then consider what we need to show:**
   We have to show that the columns of $\mathbf{A}$ are linearly dependent. Using the definition of linear dependence from **Note 3 Subsection 3.1.1**, write a mathematical equation that conveys linear dependence of columns of $\mathbf{A}$.

   (iii) **How to go from "what we know" to "what we need to show" :**
   Now manipulate the expression from (i) using mathematically logical steps to reach the expression from part (ii).

   (b) **Show that if the system of linear equations: $\mathbf{A}\vec{x} = \vec{b}$, has at least one solution for $\mathbf{A} \in \mathbb{R}^{m \times n}$, then b should be in the span of the columns of A.**

   Please also look into **Note 3 Subsection 3.3** for the definition of span.

2. **Inverse of a Matrix-Matrix Product**

   Learning Goal: This problem aims to familiarize you with the properties of inverse and related proof techniques.

   Prove that if a matrix-matrix product $\mathbf{AB}$ is invertible, the inverse will be equal to $\mathbf{B}^{-1}\mathbf{A}^{-1}$. Please see **Note 6: subsection 6.1.1** for properties of inverse.

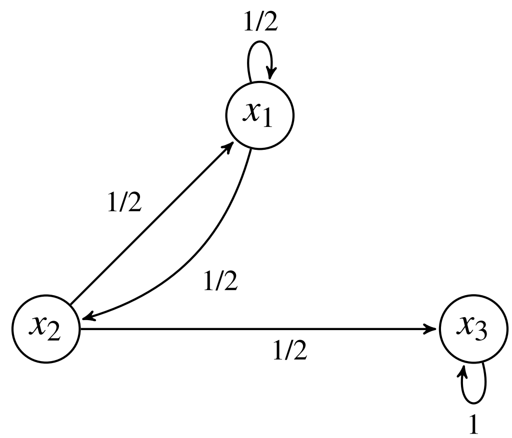
   HINT: We start again with what we know. Since $\mathbf{AB}$ is invertible, we know that an inverse exists, i.e.

$$(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}$$
$$(\mathbf{AB})^{-1}(\mathbf{AB}) = \mathbf{I}$$

3. **Functional Pumps**

   Learning Goal: The goal of this problem is to present a state transition diagram and guide students to understand the meaning of a state transition matrix and its applications. Please review **Note 5: Section 5.1** to understand this problem better.

   Take a look at this functional pump:



(a) What do the rows in a functional pump represent? What do the columns represent?

(b) Analyze the pump above and write the first column of the state transition matrix. Use the state vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Repeat this process for each of the reservoirs in this diagram.

(c) Is this system conserved? Why or why not? Please review **Note 5: Section 5.1.4** to understand this problem better.

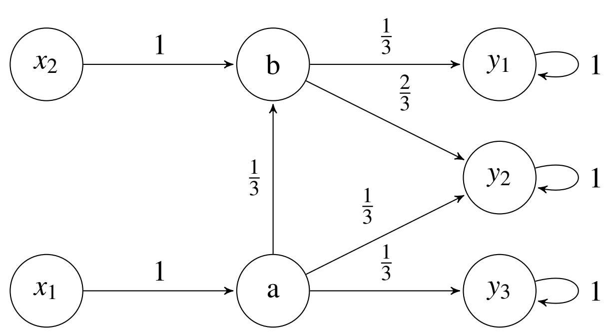
(d) Given that the initial reservoir volume, $v[0]$, is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ determine the amount of water in each of the reservoirs after turning the system on n number of times. Please review **Note 5: Section 5.1.7** to understand this problem better.

i. Turn the system on once.

ii. Turn the system on twice.

iii. What is another way to find $v[2]$ if you could only multiply one state transition matrix into the initial state once?

(e) **(PRACTICE)** Let us model a system with reservoir states $x_1, x_2, a, b, y_1, y_2, y_3$ as given by the diagram below:



Write the state transition matrix for the above state transistion diagram. Use the state vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ a \\ b \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

4. Invertibility and Row Operations

Learning Goal: This question introduces, through the context of finding a given matrix's inverse, how we can represent different types of transformations and row operations with matrices. Also, we will see whether the *order* of applying matrix operations matters. Please review **Section 2.1 of Note 2B** and **Section 6.1 of Note 6** to understand the problem better.

(a) Say we have a matrix $\mathbf{M} \in \mathbb{R}^{3\times n}$ and a matrix $\mathbf{A}$, which are given by:

$$\mathbf{M} = \begin{bmatrix} \vec{m}_1{}^T \\ \vec{m}_2{}^T \\ \vec{m}_3{}^T \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we left multiply $\mathbf{M}$ by $\mathbf{A}$ (computing the product $\mathbf{AM}$), what kind of row operation is done on $\mathbf{M}$?

(b) We have the matrix $\mathbf{M} \in \mathbb{R}^{3\times n}$ as before, as well as the matrix $\mathbf{B}$, which is given by:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

If we left-multiply $\mathbf{M}$ by $\mathbf{B}$, what kind of row operation is done on $\mathbf{M}$?

(c) We have the matrix $\mathbf{M} \in \mathbb{R}^{3\times n}$ as before, as well as the matrix $\mathbf{C}$, given by:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What kind of row operation is done on $\mathbf{M}$?

(d) What happens when we apply the transformations (row operations) described in parts (a), (b), and (c) to the matrix $\mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$?

(e) Multiply the matrices for each of the transformations in parts (a), (b), and (c), so that the are applied in this order: (a) is applied first and (c) is applied last. Call the resulting matrix $\mathbf{D}$. What happens when you left multiply the $\mathbf{Q}$ from part (d) by $\mathbf{D}$? What about right multiplying $\mathbf{Q}$ by $\mathbf{D}$? What kind of matrix is $\mathbf{D}$ in relation to $\mathbf{Q}$?

(f) Are there a set of transformations we can apply to $\mathbf{Q} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$ to make it the identity? If so, what are they? If not, why is is not possible?

