CSM 16A Designing Information Devices and Systems I
Spring 2021 Week 2

1. **Proof on Linear (In)Dependence** [WALK-THROUGH]

Learning Goal: The goal of this problem is to practice some proof development skills.

(a) **Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{0}$, has a non-zero solution, then the columns of $\mathbf{A} \in \mathbb{R}^{m\times n}$ are linearly dependent.**

We are going to use the approach outlined in **Note 4**. Please also look into **Note 3 Subsection 3.1.1** for the definition of linear dependence/ independence.

(i) **Start with what we already know:**

We know that system of equations, $\mathbf{A}\vec{x} = \vec{0}$, has a non-zero solution, $\vec{u}$. Express this information in a mathematical form.

Answer: So let us assume that $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ is a non-zero solution to $\mathbf{A}\vec{x} = \vec{0}$. So $\vec{u}$ must satisfy:

$$\mathbf{A}\vec{u} = \vec{0}, \quad \text{where} \quad \vec{u} \neq \vec{0}. \tag{1}$$

So $u_1, u_2, \ldots, u_n$ all cannot be zero, which means there is at least one $u_i \neq 0$.

(ii) **Then consider what we need to show:**

We have to show that the columns of $\mathbf{A}$ are linearly dependent. Using the definition of linear dependence from **Note 3 Subsection 3.1.1**, write a mathematical equation that conveys linear dependence of columns of $\mathbf{A}$.

Answer: Let us assume that $\mathbf{A}$ has columns $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$.

According to the definition of linear dependence:

$$\alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \ldots + \alpha_n\vec{c}_n = \vec{0}. \tag{2}$$

where not all α_i's are equal to zero.

(iii) **How to go from "what we know" to "what we need to show" :**

Now manipulate the expression from (i) using mathematically logical steps to reach the expression from part (ii).

Answer:

Since your answer to (ii) is expressed in terms of the column vectors of $\mathbf{A}$, let us try to express the mathematical equations from (i), in terms of the the column vectors too.

We can write

$$\mathbf{A}\vec{u} = \vec{0}$$

$$\implies \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix} = \vec{0}$$

$$\implies u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n\vec{c}_n = \vec{0}$$

Here not all u_i's are equal to zero. This expression matches the expression from part (ii), if we choose $\vec{u} = \vec{\alpha}$. Hence the columns of $\mathbf{A}$ are linearly dependent and the proof is complete.

(b) **Show that if the system of linear equations: $\mathbf{A}\vec{x} = \vec{b}$, has at least one solution for $\mathbf{A} \in \mathbb{R}^{m\times n}$, then b should be in the span of the columns of A.**

Please also look into **Note 3 Subsection 3.3** for the definition of span.

Answer: Start with what we already know:

We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has at least one solution. We express this information in a mathematical form.

So let us assume that $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ is a solution to $\mathbf{A}\vec{x} = \vec{b}$. So $\vec{u}$ must satisfy:

$$\mathbf{A}\vec{u} = \vec{b}. \tag{3}$$

Then consider what we need to show:

We have to show that $\vec{b}$ is in the span of the columns of $\mathbf{A}$. Using the definition of span from **Note 3 Subsection 3.3**, we write a mathematical equation to express this information.

Let us assume that $\mathbf{A}$ has columns $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \dots & | \\ \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \\ | & | & \dots & | \end{bmatrix}$. According to the definition of span:

$$\vec{b} \in \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\} \tag{4}$$

$$\implies \vec{b} = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \ldots + \alpha_n\vec{c}_n, \tag{5}$$

where α_i's are scalars.

Now go from "what we know" to "what we need to show" :

We manipulate the expression from (i) using mathematically logical steps to reach the expression from part (ii).

Since your answer to (ii) is expressed in terms of the column vectors of $\mathbf{A}$, let us try to express the mathematical equations from (i), in terms of the the column vectors too.

We can write

$$\mathbf{A}\vec{u} = \vec{b}$$

$$\implies \begin{bmatrix} | & | & \dots & | \\ \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} = \vec{b}$$

$$\implies u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n\vec{c}_n = \vec{b}$$

This expression matches the expression from part (ii), i.e. $\vec{b}$ can be expressed as a linear combination of the column vectors. So $\vec{b} \in \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$. The proof is complete.

2. Inverse of a Matrix-Matrix Product

Learning Goal: This problem aims to familiarize you with the properties of inverse and related proof techniques.

Prove that if a matrix-matrix product $\mathbf{AB}$ is invertible, the inverse will be equal to $\mathbf{B}^{-1}\mathbf{A}^{-1}$. Please see **Note 6: subsection 6.1.1** for properties of inverse.

HINT: We start again with what we know. Since $\mathbf{AB}$ is invertible, we know that an inverse exists, i.e.

$$(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}$$
$$(\mathbf{AB})^{-1}(\mathbf{AB}) = \mathbf{I}$$

Answer:

Now $\mathbf{B}^{-1}\mathbf{A}^{-1}$ will be an inverse of $\mathbf{AB}$, if and only if the following relation is satisfied:

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}.$$

For this proof, we are going to evaluate both $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$ and $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$ and see if they are equal to the identity matrix. First let us start with:

$$\begin{aligned}
(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) &= \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{AB} \\
&= \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} \\
&= \mathbf{B}^{-1}\mathbf{IB} \\
&= \mathbf{B}^{-1}(\mathbf{IB}) \\
&= \mathbf{B}^{-1}\mathbf{B} \\
&= \mathbf{I}
\end{aligned}$$

Then we evaluate:

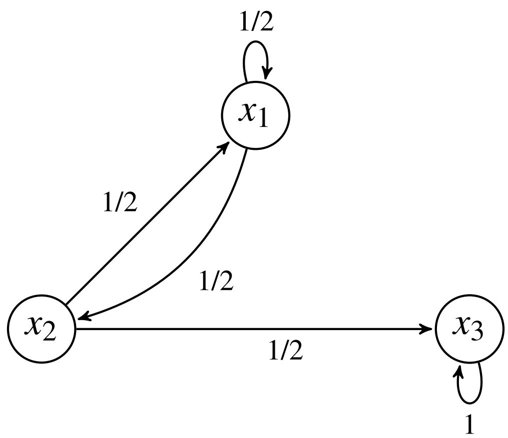
$$\begin{aligned}
(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{ABB}^{-1}\mathbf{A}^{-1} \\
&= \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} \\
&= \mathbf{A}(\mathbf{I})\mathbf{A}^{-1} \\
&= \mathbf{AA}^{-1} \\
&= \mathbf{I}
\end{aligned}$$

Since both of expressions are equal to identity matrix and the inverse for a matrix is unique, we can decide that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of $\mathbf{AB}$. So the proof is complete.

3. **Functional Pumps**

Learning Goal: The goal of this problem is to present a state transition diagram and guide students to understand the meaning of a state transition matrix and its applications. Please review **Note 5: Section 5.1** to understand this problem better.

Take a look at this functional pump:



(a) What do the rows in a functional pump represent? What do the columns represent?

Answer:

The rows in the matrix tell us how much inflow each reservoir gets from other reservoirs in the system.

The columns in the matrix tell us how much each reservoir contributes (outflow) to other reservoirs in the system. The columns also give us information about the state of the matrix. If each column of the matrix sums to 1, then the system is conserved.

To reiterate, a system being conserved means no data is lost or gained within the system.

(b) Analyze the pump above and write the first column of the state transition matrix. Use the state vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Repeat this process for each of the reservoirs in this diagram.

Answer:

$$x_1 : \begin{bmatrix} x_1 \to x_1 \\ x_1 \to x_2 \\ x_1 \to x_3 \end{bmatrix} \; x_2 : \begin{bmatrix} x_2 \to x_1 \\ x_2 \to x_2 \\ x_2 \to x_3 \end{bmatrix} \; x_3 : \begin{bmatrix} x_3 \to x_1 \\ x_3 \to x_2 \\ x_3 \to x_3 \end{bmatrix}$$
$$x_1 : \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} \; x_2 : \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} \; x_3 : \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Combining these three columns yields the state transition matrix for the above pump.

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} x_1 \to x_1 & x_2 \to x_1 & x_3 \to x_1 \\ x_1 \to x_2 & x_2 \to x_2 & x_3 \to x_2 \\ x_1 \to x_3 & x_2 \to x_3 & x_3 \to x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \end{aligned}$$

(c) Is this system conserved? Why or why not? Please review **Note 5: Section 5.1.4** to understand this problem better.

Answer: Yes; each of the columns sums up to 1. Hence, it is a conserved system.

(d) Given that the initial reservoir volume, $v[0]$, is $\begin{bmatrix}1\\1\\1\end{bmatrix}$ determine the amount of water in each of the reservoirs after turning the system on n number of times. Please review **Note 5: Section 5.1.7** to understand this problem better.

Answer: Note that turning on the system means going to the next time step.

i. Turn the system on once.

Answer:

$$\mathbf{A}v[0] = v[1]$$
$$\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 0 & 0\\0 & 1/2 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1/2\\3/2\end{bmatrix}$$
$$v[1] = \begin{bmatrix}1\\1/2\\3/2\end{bmatrix}$$

ii. Turn the system on twice.

Answer:

$$\mathbf{A}v[1] = v[2]$$
$$\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 0 & 0\\0 & 1/2 & 1\end{bmatrix}\begin{bmatrix}1\\1/2\\3/2\end{bmatrix} = \begin{bmatrix}3/4\\1/2\\7/4\end{bmatrix}$$
$$v[2] = \begin{bmatrix}3/4\\1/2\\7/4\end{bmatrix}$$

iii. What is another way to find $v[2]$ if you could only multiply one state transition matrix into the initial state once?

Answer:

$$\begin{aligned}\mathbf{A}v[1] &= v[2]\\ \implies \mathbf{A}(\mathbf{A}v[0]) &= v[2]\\ \implies \mathbf{A}^2 v[0] &= v[2]\\ \implies \mathbf{B}v[0] &= v[2],\end{aligned}$$

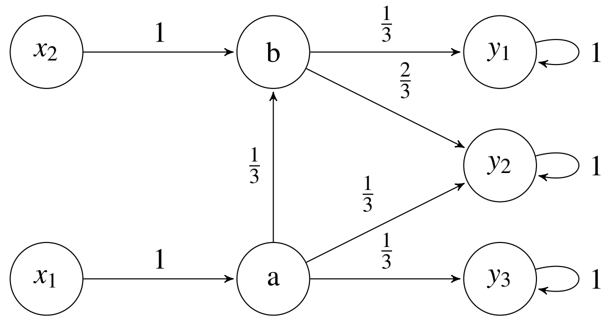
where $\mathbf{A}^2 = \mathbf{B}$

$$\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 0 & 0\\0 & 1/2 & 1\end{bmatrix}\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 0 & 0\\0 & 1/2 & 1\end{bmatrix} = \begin{bmatrix}1/2 & 1/4 & 0\\1/4 & 1/4 & 0\\1/4 & 1/2 & 1\end{bmatrix} = \mathbf{B}$$
$$\mathbf{B}v[0] = \begin{bmatrix}1/2 & 1/4 & 0\\1/4 & 1/4 & 0\\1/4 & 1/2 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3/4\\1/2\\7/4\end{bmatrix}$$

Similarly we can deduce that:

$$\mathbf{A}^n v[0] = v[n]$$

(e) **(PRACTICE)** Let us model a system with reservoir states $x_1, x_2, a, b, y_1, y_2, y_3$ as given by the diagram below:



Write the state transition matrix for the above state transistion diagram. Use the state vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ a \\ b \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Invertibility and Row Operations

Learning Goal: This question introduces, through the context of finding a given matrix's inverse, how we can represent different types of transformations and row operations with matrices. Also, we will see whether the *order* of applying matrix operations matters. Please review **Section 2.1 of Note 2B** and **Section 6.1 of Note 6** to understand the problem better.

(a) Say we have a matrix $\mathbf{M} \in \mathbb{R}^{3\times n}$ and a matrix $\mathbf{A}$, which are given by:

$$\mathbf{M} = \begin{bmatrix} \vec{m}_1{}^T \\ \vec{m}_2{}^T \\ \vec{m}_3{}^T \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If we left multiply $\mathbf{M}$ by $\mathbf{A}$ (computing the product $\mathbf{AM}$), what kind of row operation is done on $\mathbf{M}$?

Answer: Given any matrices $\mathbf{P}$ and $\mathbf{Q}$, where $\mathbf{Q}$ is written as "stacked" row vectors, the product $\mathbf{PQ}$ is:

$$\mathbf{PQ} = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{bmatrix} \begin{bmatrix} \vec{q_1}^T \\ \vec{q_2}^T \\ \vec{q_3}^T \end{bmatrix} = \begin{bmatrix} p_{1,1}\vec{q_1}^T + p_{1,2}\vec{q_2}^T + p_{1,3}\vec{q_3}^T \\ p_{2,1}\vec{q_1}^T + p_{2,2}\vec{q_2}^T + p_{2,3}\vec{q_3}^T \\ p_{3,1}\vec{q_1}^T + p_{3,2}\vec{q_2}^T + p_{3,3}\vec{q_3}^T \end{bmatrix}$$

The left matrix's constants $p_{i,j}$ all follow the same pattern: for each row (i), there is one 1 entry and the rest are 0's. This tells us where rows will appear in the final product. For instance, R_1 of $\mathbf{P}$ is $\begin{bmatrix} p_{1,1} = 0 & p_{1,2} = 0 & p_{1,3} = 1 \end{bmatrix}$, so the product's first row will be $\vec{m_3}^T$.

Applying this throughout, the product $\mathbf{AM} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix} = \begin{bmatrix} 1 * \vec{m_1}^T \\ \frac{1}{5} * \vec{m_2}^T \\ 1 * \vec{m_3}^T \end{bmatrix}$.

So we end up with a matrix where the first and third rows are identical to $\mathbf{M}$, but the second row is $\frac{1}{5} * \vec{m_2}^T$. So the row operation performed here is scaling R_2 by a fifth.

More generally, if $\mathbf{M} = \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ then $AM = \begin{bmatrix} a_1\vec{m_1}^T \\ a_2\vec{m_2}^T \\ a_3\vec{m_3}^T \end{bmatrix}$. $\mathbf{A}$ is transforming $\mathbf{M}$ by scaling each of its rows by a_1, a_2, and a_3, respectively. Also interesting to note: $\mathbf{A}$ is the result after you perform the same scaling row operations on the identity matrix!

(b) We have the matrix $\mathbf{M} \in \mathbb{R}^{3\times n}$ as before, as well as the matrix $\mathbf{B}$, which is given by:

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

If we left-multiply $\mathbf{M}$ by $\mathbf{B}$, what kind of row operation is done on $\mathbf{M}$?

Answer: Refer to the solution for part (a) to revisit the concept of "stacked" row vectors.

Applying this concept throughout gives $\mathbf{BM} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix} = \begin{bmatrix} \vec{m_3}^T \\ \vec{m_2}^T \\ \vec{m_1}^T \end{bmatrix}$. So the resulting product matrix has rows in order: $\vec{m_3}^T, \vec{m_2}^T, \vec{m_1}^T$. This indicates that the row operation that was done on $\mathbf{M}$ is swapping rows one and three.

(c) We have the matrix $\mathbf{M} \in \mathbb{R}^{3\times n}$ as before, as well as the matrix $\mathbf{C}$, given by:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What kind of row operation is done on $\mathbf{M}$?

Answer: We use the same method as the solution in part (a), except this time we end up with:

$$\mathbf{CM} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{m_1}^T \\ \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix} = \begin{bmatrix} \vec{m_1}^T \\ 3\vec{m_1}^T + \vec{m_2}^T \\ \vec{m_3}^T \end{bmatrix}$$

which shows that the row operation done on $\mathbf{M}$ was adding $3 * R_1$ to R_2.

(d) What happens when we apply the transformations (row operations) described in parts (a), (b), and (c) to the matrix $\mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$?

Answer: If we scale row 2 by $1/5$, then swap row 1 and row 3, then add 3 times row 1 to row 2, we get the identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) Multiply the matrices for each of the transformations in parts (a), (b), and (c), so that the are applied in this order: (a) is applied first and (c) is applied last. Call the resulting matrix $\mathbf{D}$. What happens when you left multiply the $\mathbf{Q}$ from part (d) by $\mathbf{D}$? What about right multiplying $\mathbf{Q}$ by $\mathbf{D}$? What kind of matrix is $\mathbf{D}$ in relation to $\mathbf{Q}$?

Answer:

$$\mathbf{D} = \mathbf{CBA} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{5} & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

Left multiplying $\mathbf{Q}$ by $\mathbf{D}$:

$$\mathbf{DQ} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{5} & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Right multiplying $\mathbf{Q}$ by $\mathbf{D}$:

$$\mathbf{QD} = \begin{bmatrix} 0 & 0 & 1 \\ -15 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{5} & 3 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\mathbf{QD} = \mathbf{DQ} = \mathbf{I}$, $\mathbf{D}$ is the inverse of $\mathbf{Q}$, often written as $\mathbf{Q}^{-1}$.

(f) Are there a set of transformations we can apply to $\mathbf{Q} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$ to make it the identity? If so, what are they? If not, why is is not possible?

Answer:

To make it the identity matrix, we use Gaussian Elimination:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right].$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

There is another way to approach this question, graphically. When we multiply a matrix A with another matrix X in the form $AX = B$, we are performing a linear transformation on X and shifting it to the A's "coordinate system." This means the transformed matrix B must be in the span of the A. Thus, if A is of lower dimension than the X, B loses a dimension. In essence, it loses information that we cannot retrieve again with an inverse matrix. An example of this is if we shine a light on an apple (3D) and look at its shadow (2D). There is no way to reconstruct the 3D apple as we have irreparably lost information in the transformation.

