CSM 16A Spring 2021 — Designing Information Devices and Systems I — Week 11

1. **Properties of Norms**
Prove each of the following theorems using definitions and properties of the inner product and orthogonality.

(a) **Pythagoras Theorem**
Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal set of vectors (meaning $\vec{v}_i \perp \vec{v}_j \forall i \neq j$),

$$\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 + \cdots + \|\vec{v}_n\|^2$$

Answer:

$$\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\|^2 =< \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n, \vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n >$$
$$= \sum_{i=1}^{n} < \vec{v}_i, \vec{v}_i > + \sum_{i \neq j} < \vec{v}_i, \vec{v}_j >$$

Since it follows that $\vec{v}_i \perp \vec{v}_j \forall i \neq j$, we know that $< \vec{v}_i, \vec{v}_j >= 0 \, \forall i \neq j$. Hence, the above expression simplifies to:

$$= \sum_{i=1}^{n} < \vec{v}_i, \vec{v}_i >= \sum_{i=1}^{n} \|\vec{v}_i\|^2$$

(b) **The Cauchy-Schwarz Inequality**
For any vectors $\vec{u}$ and $\vec{v}$,

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Answer: If $\vec{u}$ or $\vec{v}$ equals $\vec{0}$, then both sides of the inequality equal 0, and $0 \leq 0$ holds. However, if $\vec{u}$ and $\vec{v}$ are both nonzero, we know that

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

We also know that $|\cos\theta| \leq 1$ and $\|\vec{u}\| \|\vec{v}\| \geq 0$, so if we take the absolute value of the previous equation, we get

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\| |\cos\theta|$$
$$\leq \|\vec{u}\| \|\vec{v}\|$$

(c) **Triangle Inequality**
For any vectors $\vec{u}$ and $\vec{v}$,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Answer: Square both sides of this inequality:

$$(\|\vec{u} + \vec{v}\|)^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$$

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Let's expand the left side of the equation:

$$\begin{aligned}(\|\vec{u}+\vec{v}\|)^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})\\ &= \vec{u}\cdot(\vec{u}+\vec{v})+\vec{v}\cdot(\vec{u}+\vec{v})\\ &= \vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v}\\ &= \|\vec{u}\|^2+2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2\end{aligned}$$

Now the right side:

$$\begin{aligned}(\|\vec{u}\|+\|\vec{v}\|)^2 &= (\|\vec{u}\|+\|\vec{v}\|)(\|\vec{u}\|+\|\vec{v}\|)\\ &= \|\vec{u}\|(\|\vec{u}\|+\|\vec{v}\|)+\|\vec{v}\|(\|\vec{u}\|+\|\vec{v}\|)\\ &= \|\vec{u}\|\|\vec{u}\|+\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|\|\vec{u}\|+\|\vec{v}\|\|\vec{v}\|\\ &= \|\vec{u}\|^2+2\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|^2\end{aligned}$$

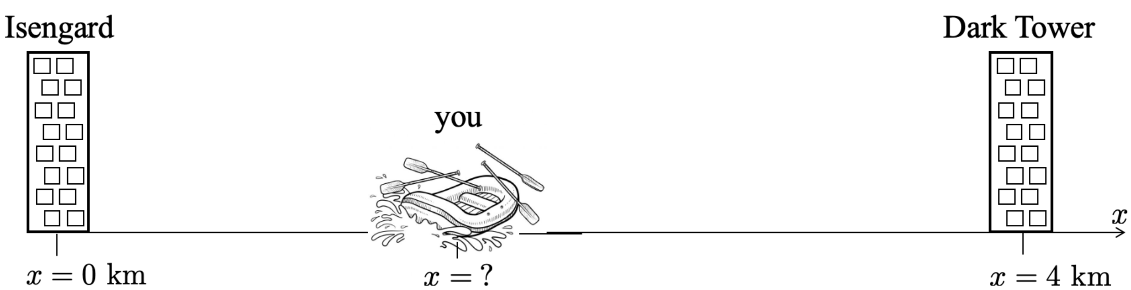
Now, from Cauchy-Schwarz, we know that the middle term of the left side $2|\vec{u}\cdot\vec{v}|$ is less than the middle term of the right side $2\|\vec{u}\|\|\vec{v}\|$. Thus, the Triangle Inequality holds. (Note: single vertical bars means absolute value, and double vertical bars means magnitude.)

2. **One Does Not Simply Raft into Mordor**

Learning Goal: The goal of this problem is to practice applying cross correlation to word problem scenarios.

Relevant Notes: **Note 22 Section 22.2** covers trilateration, and **Note 22 Section 22.3** covers finding distances with correlation.

You've decided to go rafting to celebrate taking your second midterm! Unfortunately, an hour into your trip, you realize that there are no familiar landmarks nearby, so you're not sure how far you are from your starting point. However, you do remember from your studies of the area that there are two towers: Isengard, at the position $x = 0$ km, and the Dark Tower, at $x = 4$ km. You know you are between the two towers, as shown below:



You recall that each tower emits a sound signal once a day at midday. Specifically, Isengard will emit $\vec{b}_1$ and the Dark Tower will emit $\vec{b}_2$:

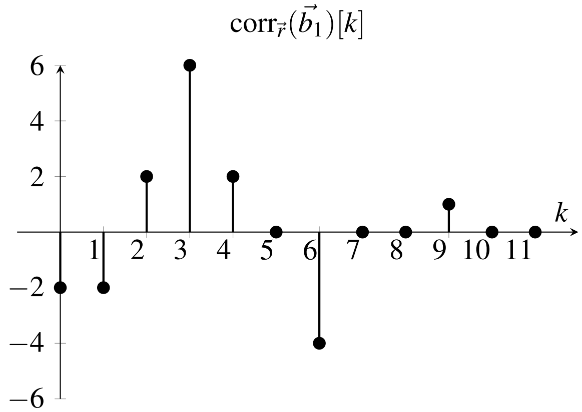
$$\vec{b}_1 = \begin{bmatrix}-1 & -1 & -1 & 1 & 1\end{bmatrix}^T \qquad \vec{b}_2 = \begin{bmatrix}-1 & 1 & -1 & 1 & -1\end{bmatrix}^T$$

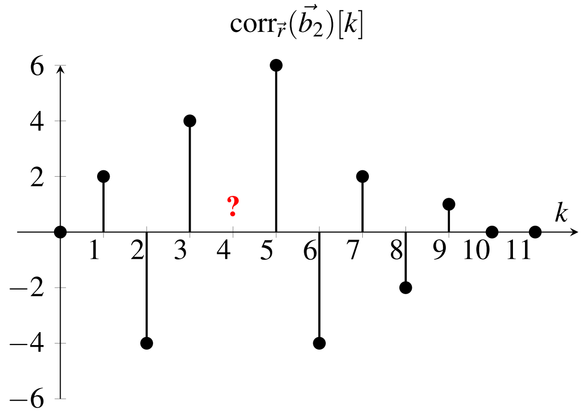
Both signals are emitted at a rate of 2 samples per second (i.e. the sample interval is 0.5 sec), and the signals are emitted only once for each period.

It's only a few minutes from midday so you decide to wait. You use an app on your phone to record the incoming signal (the app also records at 2 samples per second). You start recording at exactly 12:00 PM and receive the following:

$$\vec{r} = \begin{bmatrix}0 & 0 & 0 & -1 & -1 & -2 & 2 & 0 & 1 & -1 & 0 & 0\end{bmatrix}^T$$

(a) Your first step is to calculate a linear cross-correlation between $\vec{r}$ and each known tower signature. The cross-correlations are plotted below. Calculate the missing value, which is denoted with a question mark.





Answer: To fill in the missing cross-correlation values, observe that the missing value is in $\text{corr}_{\vec{r}}(\vec{b_2})$ is at index 4. Therefore, we must calculate the inner product between $\vec{r}$ and a version of $\vec{b_2}$ that is shifted 4 samples to the right.

$$\vec{r} = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & -2 & 2 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$
$$\vec{b}_{2,\text{shifted by }4} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & \end{bmatrix}$$

We calculate:

$$\begin{aligned} \text{corr}_{\vec{r}}(\vec{b_2})[4] =& (-1)(0)+(-1)(-1)+(-2)(1)+(2)(-1)+(0)(1)+(1)(-1)+(-1)(0) \\ =& 1-2-2+0-1 \\ =& -4 \end{aligned}$$

(b) Recall that the signals were emitted from each building at the same time (12:00 PM). How many seconds after 12:00 PM did it take for the signal from Isengard to reach you? What about the signal from the Dark Tower? Assume that environmental noise (besides the tower-emitted signals) is minimal.

Answer: From the completed correlation diagrams solved for in (a), we find that $\text{corr}_{\vec{r}}(\vec{b_1})$ is maximized at sample 3 (value 6) and $\text{corr}_{\vec{r}}(\vec{b_2})$ is maximized at sample 5 (value 6). Note that each maximum isn't exactly equal to the norm of the respective tower's signal squared because there is noise in this problem. The time needed for the app to measure 3 samples is $\frac{3\text{ samples}}{2\text{ samples / sec}} = 1.5$ sec. Similarly, the time needed for the app to measure 5 samples is $\frac{5\text{ samples}}{2\text{ samples / sec}} = 2.5$ sec. Hence, the number of seconds it took for the signal from Isengard to reach you was 1.5 sec. From the Dark Tower, it took 2.5 sec seconds.

(c) Now, assume that you received Isengard's signal 8 seconds after it was sent and you received the Dark Tower's signal 2 seconds after it was sent. Can you determine your exact position x? If yes, calculate your position. If not, explain why not. Assume sound travels at 340 m/s.

Answer: Sound travels at 340 m/s, so the measurement from Isengard tells you that you are 8 sec $\times$ 340 m/sec $=$ 2,720 m $=$ 2.72 km from Isengard.

Similarly, the measurement from the Dark Tower tells you that you are 2 sec $\times$ 340 m/sec $=$ 680 m $=$ 0.68 km from Isengard.

You know that the two towers are 4 km apart. Therefore the measurement from Isengard indicates that you are at $x = 2.72$ km and the measurement from the Dark Tower indicates that you are at $x = 4$

km - 0.68 km = 3.32 km. These measurements are inconsistent, so you cannot determine your exact position.

(d) You see a giant eagle, so you get out of your raft to follow it. But you soon realize that you don't know your x position *or* your y position! Luckily, you have a phone app which tells you that you are:

- d_1 km away from Isengard which is located at $x = 0$ km, $y = 0$ km
- d_2 km away from the Dark Tower which is located at $x = 4$ km, $y = 0$ km
- d_3 km away from Minas Tirith which is located at $x = 1$ km, $y = 3$ km

Write a system of linear equations of the form $\mathbf{A}\vec{x} = \vec{b}$ that you can solve to find your position. Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ where x, y have units of kilometers (km).

Answer:
Let $\vec{a}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ be the position of the Dark Tower and $\vec{a}_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ be the position of Minas Tirith.
The three towers give us the following equations:

- Isengard: $\|\vec{x}\|^2 = d_1^2$
- The Dark Tower: $\|\vec{x} - \vec{a}_2\|^2 = d_2^2$
- Minas Tirith: $\|\vec{x} - \vec{a}_3\|^2 = d_3^2$

Rewriting these using transpose notation we get:

$$\vec{x}^T\vec{x} = d_1^2 \tag{1}$$
$$\vec{x}^T\vec{x} - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = d_2^2 \tag{2}$$
$$\vec{x}^T\vec{x} - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = d_3^2 \tag{3}$$

We subtract equation 2 from equation 1, and separately we subtract equation 3 from equation 1. Then we get:

$$2\vec{a}_2^T\vec{x} - \|\vec{a}_2\|^2 = d_1^2 - d_2^2$$

and

$$2\vec{a}_3^T\vec{x} - \|\vec{a}_3\|^2 = d_1^2 - d_3^2$$

These two equations are linear in $\vec{x}$, write them in matrix-vector form:

$$\begin{bmatrix} 2\vec{a}_2^T \\ 2\vec{a}_3^T \end{bmatrix} \vec{x} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \end{bmatrix}.$$

We see that

$$\mathbf{A} = \begin{bmatrix} 2\vec{a}_2^T \\ 2\vec{a}_3^T \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \end{bmatrix}.$$

Plugging in the values of $\vec{a}_2$ and $\vec{a}_3$ gives:

$$\mathbf{A} = \begin{bmatrix} 8 & 0 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 16 + d_1^2 - d_2^2 \\ 10 + d_1^2 - d_3^2 \end{bmatrix}.$$

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3. Positioning with Gold Code

Learning Goal: The goal of this problem is to understand how GPS signals are encoded and decoded.

Relevant Notes: **Note 22: Sections 22.3-22.5** walk through the math behind correlations, leading to an overview of the positioning problem.

For this problem, we assume there are 4 GPS satellites in total (In reality, the actual GPS system uses at least 24). Each satellite uses a unique 1024 element long sequence called a Gold code as its "signature." Assume the four satellites use the signatures: $\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$ and $\vec{s}_4$.

(a) The GPS device receives a 3575 element long signal $\vec{r}$ spanning $n = 0 - 3574$. If we wanted to identify which satellites' signatures are present in $\vec{r}$, what specific correlations should we calculate?

Hint: Recall that the linear correlation of signal $\vec{x}$ with signal $\vec{y}$ is given as:

$$corr_{\vec{x}}(\vec{y})[k] = \sum_{n=-\infty}^{\infty} \vec{x}[n]\vec{y}[n-k]$$

.

Answer: We need to calculate the correlation of received signal $\vec{r}$ with all four satellite signatures in order to find which ones are present in $\vec{r}$, i.e. we need to calculate

- $\text{corr}_{\vec{r}}(\vec{s}_1)$
- $\text{corr}_{\vec{r}}(\vec{s}_2)$
- $\text{corr}_{\vec{r}}(\vec{s}_3)$
- $\text{corr}_{\vec{r}}(\vec{s}_4)$

(b) For each of the correlation operations from part (a), what should be the finite range of time shift k? Remember that $\vec{r}$ has 3575 elements.

Answer: For each correlation operation the received signal $\vec{r}$ stays stationary while the Gold code s_i is shifted.

The range of k is chosen such that there is some overlap between $\vec{r}$ and the shifted signal $s_1[n-k]$. Hence the starting value of k will be chosen such that the last sample from $\vec{s}_1[n-k]$ overlaps with the first sample of $\vec{r}$. Since $\vec{r}$ starts at $n = 0$, this will happen when $\vec{s}_1$ will be advanced by 1023 time samples (or you could think of it as being "delayed" by -1023 time samples).

Similarly, the end value of k will be chosen such that the first sample from $\vec{s}_1[n-k]$ overlaps with the last sample of $\vec{r}$. Since $\vec{r}$ ends at $n = 3574$, this will happen when $\vec{s}_1$ will be delayed by 3574 time samples.

So the range of time shift will be $k = -1023$ to 3574.

(c) Each satellite signature is a sequence of 1024 elements, where each element is either $+1$ or -1. Assuming there is no noise, what would be approximate peak value of $\text{corr}_{\vec{r}}(\vec{s}_1)$, if $\vec{s}_1$ and $\vec{s}_2$ both are present in $\vec{r}$?

Answer: If $\vec{s}_1$ and $\vec{s}_2$ are present in $\vec{r}$, we can write $\vec{r}$ as

$$\vec{r}[n] = \vec{s}_1[n-n_{d1}] + \vec{s}_2[n-n_{d2}],$$

where n_{d1} and n_{d2} are the transmission delays of $\vec{s}_1$ and $\vec{s}_2$ receptively. So $\vec{r}$ perfectly matches $\vec{s}_1$ when $\vec{s}_1$ is shifted by n_{d1} samples, i.e. peak correlation will occur. If we calculate the correlation $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$

for time shift $k = n_{d1}$, we have

$$\begin{aligned}
\text{corr}_{\vec{r}}(\vec{s}_1)[k = n_{d1}] &= \sum_{n=-\infty}^{\infty} \vec{r}[n]\vec{s}_1[n - n_{d1}] && (4)\\
&= \sum_{n=-\infty}^{\infty} (\vec{s}_1[n - n_{d1}] + \vec{s}_2[n - n_{d2}])\vec{s}_1[n - n_{d1}] && (5)\\
&= \langle \vec{s}_1[n - n_{d1}], \vec{s}_1[n - n_{d1}] \rangle + \langle \vec{s}_2[n - n_{d2}], \vec{s}_1[n - n_{d1}] \rangle && (6)
\end{aligned}$$

The first term can be reduced to

$$\langle \vec{s}_1[n - n_{d1}], \vec{s}_1[n - n_{d1}] \rangle = \|\vec{s}_1\|^2 = (\sqrt{1024})^2 = 1024$$

while the second term can be reduced to

$$\langle \vec{s}_2[n - n_{d2}], \vec{s}_1[n - n_{d1}] \rangle \approx 0$$

Hence $\text{corr}_{\vec{r}}(\vec{s}_1)[k = n_{d1}] \approx 1024$, i.e. the peak correlation value will be 1024.

Another intuitive way of thinking about this problem is that if the signal matches perfectly, then the inner product will just be a sum of 1024 ones. However, if there are a lot of mismatches then there will be a lot of terms that sum to zero. Note that the inner product of non-corresponding signals will be more in a range of what looks like -200 to 200 on the diagram because there will be some random matches (though it should average out to zero).

(d) The following figure shows the results of cross-correlation of the received signal $\vec{r}$ with respect to different satellite signatures.

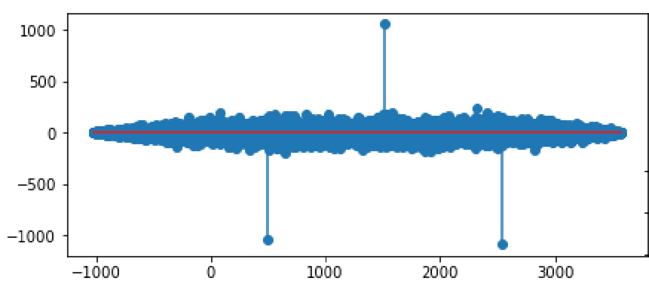

Figure 1: $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ vs. k

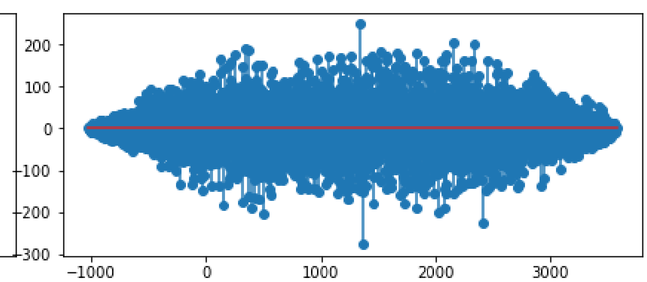

Figure 2: $\text{corr}_{\vec{r}}(\vec{s}_2)[k]$ vs. k

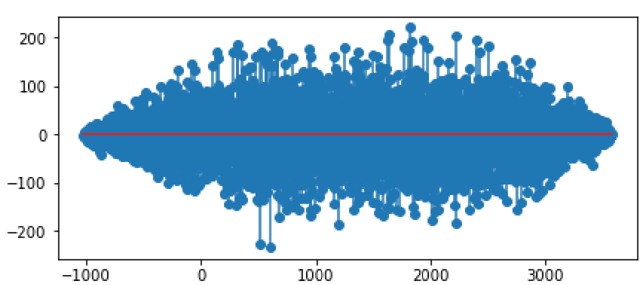

Figure 3: $\text{corr}_{\vec{r}}(\vec{s}_3)[k]$ vs. k

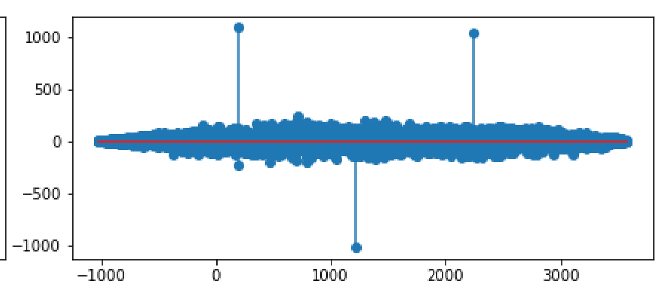

Figure 4: $\text{corr}_{\vec{r}}(\vec{s}_4)[k]$ vs. k

Find out which satellite signals are present and the corresponding transmission delays. Assume a threshold of 800 for determining the peak correlation.

Answer: From the correlation plots, we observe that $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{r}}(\vec{s}_4)[k]$ have positive/negative peaks that exceed the threshold value. Hence we can determine that $\vec{s}_1$ and $\vec{s}_4$ are present in the received signal.

The position of the first peak corresponds to the transmission delay. From the plot we see the first peak of $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ occur at $k \approx 500$, hence the delay for $\vec{s}_1$ is $n_{d1} \approx 500$ time samples. Similarly, the delay for $\vec{s}_4$ is $n_{d4} \approx 100$ time samples.

(e) Assume consecutive time samples are $\delta t = 0.1\mu\text{s} = 0.1 \times 10^{-6}\text{s}$ apart. Use your results from the last part to find out distance of satellites from the receiver.

Answer: The transmission delay t_{d1} for $\vec{s}_1$ can be calculated in seconds using:

$$t_{d1} = n_{d1} \times \delta t$$

$$t_{d1} = 500 \times 0.1\mu\text{s} = 50\mu\text{s} = 5 \times 10^{-5}\text{s}.$$

Similarly, transmission delay t_{d4} for $\vec{s}_4$ is:

$$t_{d4} = n_{d4} \times \delta t$$

$$t_{d4} = 100 \times 0.1\mu\text{s} = 10\mu\text{s} = 10^{-5}\text{s}.$$

Now we can find the distances by multiplying the velocity of the signals with the time delay:

$$d_1 = v \times t_{d1} = 3 \times 10^8 \times 5 \times 10^{-5} = 15\text{km}$$

$$d_4 = v \times t_{d4} = 3 \times 10^8 \times 10^{-5} = 3\text{km}$$

Take note that n_{d1} is the number of time samples that make up the delay. The time is sampled every δt seconds, (i.e., sec/sample). When you multiply δt by the number of time samples, you get the duration of the delay in seconds.

(f) In addition to sending a unique signature signal, a GPS satellite can also "modulate" the signature to communicate more information. Modulating a signature means multiplying the entire signature block by $+1$ or -1.

For example satellite 1 can transmit the sequence $\vec{s}_1$ when it wants send a message bit of 1 or $-\vec{s}_1$ when it wants to send a message bit of -1. Using the correlation plots, find out the message bits the satellites are sending.

Answer: When a receiver receives a signal, in addition to finding the time delay between transmission and reception, the receiver will be able to decode the message by noting a very high correlation if the message bit is equal to 1, and a very negative correlation if the message bit is equal to -1. From the figures we can see that satellite 1 and 4 are transmitting the following bits:
Satellite 1 message: $\{-1, 1, -1\}$
Satellite 4 message: $\{1, -1, 1\}$