EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 9

This homework is due Friday, Nov 4, 2022 at 23:59.
Self-grades are due Monday, Nov 7, 2022 at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw9.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

   For this homework, please read Notes 16, 17 (17.1 - 17.2, specifically) and 17B. Note 16 will provide an introduction to capacitors (a circuit element which stores charge), capacitive equivalence, and the underlying physics behind them. Sections 17.1 - 17.2 in Note 17 will provide an overview of the capactive touchscreen and how to measure capacitance. Note 17B will provide a walkthrough of the charge-sharing algorithm.

   (a) Consider the capacitive touchscreen. Describe how it works, and compare and contrast it to the resistive touchscreens we have seen in previous lectures and homeworks.

   (b) In the charge sharing algorithm, what property of charge is applied in connecting phase 1 calculations to phase 2 calculations?

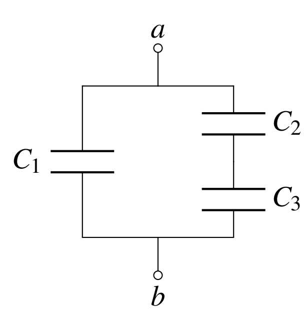
   Solution:

   a) The capacitive touch screen works by detecting a change in capacitance, which is caused by the additional capacitance of a finger being added to the capacitance of the touch screen. The resistive touch screen detects the position of a touch by modeling the touch screen as a voltage divider when pressed down.

   b) We apply the conservation of charge at floating nodes to relate measurements in different phases.

2. **Equivalent Capacitance (9 points)**

   (a) (4 points) Find the equivalent capacitance between terminals a and b of the following circuit in terms of the given capacitors C_1, C_2, and C_3. Leave your answer in terms of the addition, subtraction, multiplication, and division operators **only**.



Solution:

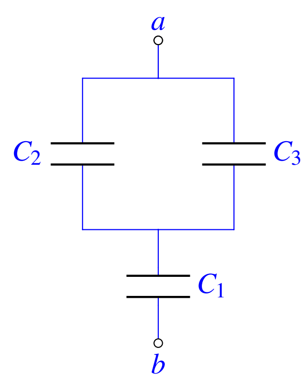
$$C_{eq} = C_1 + (C_2||C_3)$$
$$C_{eq} = C_1 + \frac{C_2C_3}{C_2+C_3}$$

Here, $||$ represents the mathematical parallel operator ($a||b = \frac{ab}{a+b}$).

(b) (5 points) Find and draw a capacitive circuit using three capacitors, C_1, C_2, and C_3, that has equivalent capacitance of

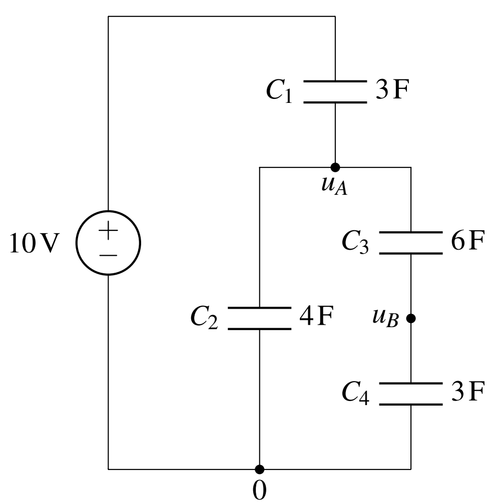
$$\frac{C_1(C_2+C_3)}{C_1+C_2+C_3}$$

Solution: This expression is the same as $C_1||(C_2+C_3)$, so C_2 and C_3 are in parallel with each other, and C_1 is series with both of them:



3. Circuit with Capacitors

Find the voltages at nodes u_A and u_B, and currents flowing through all of the capacitors at steady state. Assume that before the voltage source is applied, the capacitors all initially have a charge of 0 Coulombs.



Solution: Guide: In general, your strategy to solve circuits with capacitors should be similar to solving resistive circuits. For capacitive circuits we often care about steady state (i.e. what happens to the circuit after a long time and no more changes are happening). If we consider a circuit with capacitors and voltage sources, we will always think about steady state (or the steady state for a phase if we work on a charge sharing problem with switches). When thinking about steady state, you always want to write out the equations for charge that you know, as well as all the KVL type relationships around voltages you know. Then use the key idea that charge is conserved to build out your system of equations. Don't be daunted by the variable names and know that everything just boils down to a system of linear equations.

Here are some principles that are also helpful:

(a) Charge at a node from which charge cannot escape or enter (floating node) is always conserved. — if the sum of charges is 0 on a floating node, the sum of charges on that floating node at steady state will be zero.

(b) The charge Q stored in a capacitor is given by the equation $Q = CV$.That is, the plate that corresponds to the "+" terminal, stores $+Q = +CV$, and the plate that corresponds to the "-" terminal, stores $-Q = -CV$.

(c) If two capacitors are initially uncharged, and then are connected in series, the charges on both capacitors are equal to each other at steady state.

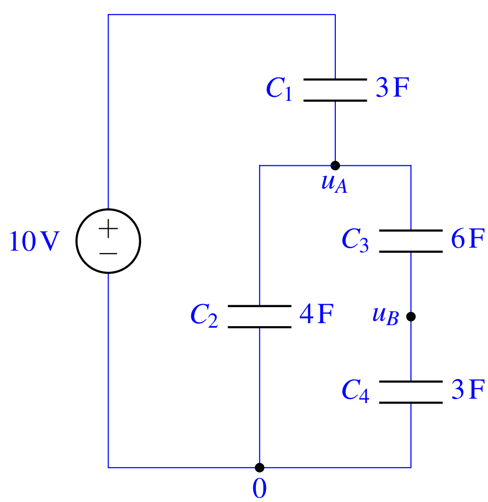
(d) The voltage across capacitors in parallel is equal at steady state.

Method 1: Charge conservation

For a capacitor C_k, let us denote the voltage across it by v_{C_k}, the current flowing through it by i_{C_k}, and its charge by Q_{C_k}. In steady state (that is, after the current has been running for a very long time), direct current (DC) capacitors act as open circuits. Hence, we see that there is no current flowing through the capacitors, that is,

$$i_{C_1} = i_{C_2} = i_{C_3} = i_{C_4} = 0\,\text{A}.$$

To find the voltages across the capacitors, let us label nodes on the circuit as shown in the following figure.



We are going to use the following four properties to find the voltages across the capacitors:

(a) Charge is always conserved at floating nodes.

(b) The charge Q stored in a capacitor is given by the equation $Q = CV$.

(c) The charges across series capacitors that are initially uncharged are equal to each other.

(d) The voltage across parallel capacitors is equal.

As an example use of property (c), we have the charge on the capacitor C_3 equal to the charge on the capacitor C_4.

Let us start by writing the equation for conservation of charge at node u_A:

$$Q_{C_1} = Q_{C_2} + Q_{C_3}$$

By property (b), that is, $Q = CV$, we can equivalently write this equation for charge conservation in terms of node voltages as

$$(10\,\text{V} - u_A)3\,\text{F} = (u_A - 0)4\,\text{F} + (u_A - u_B)6\,\text{F},$$

which, after simplifying the equation, gives

$$30\,\text{V} = 13u_A - 6u_B. \tag{1}$$

Let us then write the charge conservation equation at node u_B; we have

$$Q_{C_3} = Q_{C_4}.$$

As before, we can write this charge conservation equation in terms of the node voltages as

$$(u_A - u_B)6\,\text{F} = u_B 3\,\text{F},$$

which, after simplification, gives

$$2u_A = 3u_B. \tag{2}$$

Equations 1 and 2 give us two linearly independent equations in two unknowns. Solving the system, we get

$$u_A = 10/3\,\text{V},$$
$$u_B = 20/9\,\text{V}.$$

Using the node voltages, we can calculate the voltages across the capacitors as

$$v_{C_1} = 10\,\text{V} - u_A = 20/3\,\text{V},$$
$$v_{C_2} = u_A = 10/3\,\text{V},$$
$$v_{C_3} = u_A - u_B = 10/9\,\text{V},$$
$$v_{C_4} = u_B = 20/9\,\text{V}.$$

We write the currents across the capacitors again here for reader's convenience:

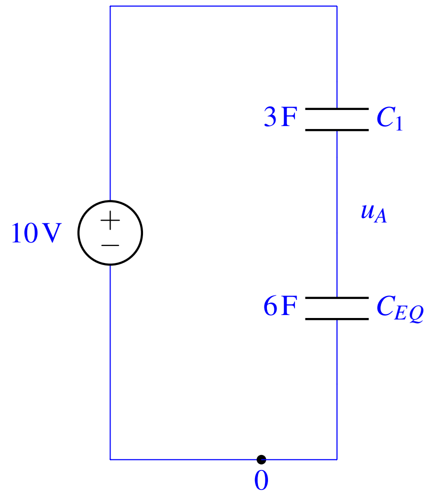
$$i_{C_1} = i_{C_2} = i_{C_3} = i_{C_4} = 0\,\text{A}$$

Method 2: Capacitor equivalence

Let's try to consider another method of solving this. We know that, initially, all the capacitors have charges of 0 C. After a 10 V voltage source is applied, the intermediate node potentials u_A and u_B will settle to some steady-state value.

Note that capacitor voltage division only works here because we know the initial conditions of the capacitors before and after the 10 V voltage source is applied. Capacitor voltage division is really just another way of solving for charge redistribution.

Let's try to find the node potential u_A.



Note that we replaced the capacitors below node u_A with an equivalent capacitance $C_{EQ} = (6\,\mathrm{F} \parallel 3\,\mathrm{F}) + 4\,\mathrm{F} = 6\,\mathrm{F}$. The equation for u_A uses the *capacitor* voltage division formula:

$$u_A = 10\,\mathrm{V}\frac{3\,\mathrm{F}}{3\,\mathrm{F}+6\,\mathrm{F}} = 10/3\,\mathrm{V}$$

We can then recognize that the potential u_B is the capacitor voltage division of u_A, namely:

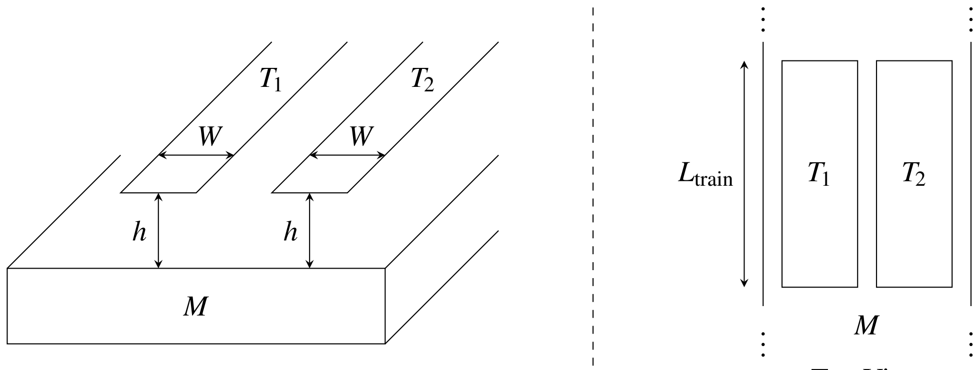
$$\begin{aligned} u_B &= \frac{6\,\mathrm{F}}{6\,\mathrm{F}+3\,\mathrm{F}} u_A \\ &= (\frac{2}{3})10/3\,\mathrm{V} \\ &= 20/9\,\mathrm{V} \end{aligned}$$

Note that these are the same values we found using Method 1.

4. Maglev Train Height Control System

One of the fastest forms of land transportation are trains that actually travel slightly elevated from the ground using magnetic levitation (or "maglev" for short). Ensuring that the train stays at a relatively constant height above its "tracks" (the tracks in this case are what provide the force to levitate the train and propel it forward) is critical to both the safety and fuel efficiency of the train. In this problem, we'll explore how maglev trains use capacitors to stay elevated. (Note that real maglev trains may use completely different and much more sophisticated techniques to perform this function, so if you get a contract to build such a train, you'll probably want to do more research on the subject.)

(a) As shown below, we put two parallel strips of metal (T_1, T_2) along the bottom of the train and we have one solid piece of metal (M) on the ground below the train (perhaps as part of the track).



Side View Top View

Assuming that the entire train is at a uniform height above the track and ignoring any fringing fields (i.e., we can use the simple equations developed in lecture to model the capacitance), as a function of L_{train} (the length of the train), W (the width of T_1 and T_2), and h (the height of the train off of the track), what is the capacitance between T_1 and M? What is the capacitance between T_2 and M?

Solution:

The distance between the plates (T_1 & M or T_2 & M) is h. The area of the parallel plate capacitor is $A = WL_{\text{train}}$. Using the formula for capacitance of a parallel plate capacitor, we get:

$$C = \frac{\varepsilon A}{d}$$

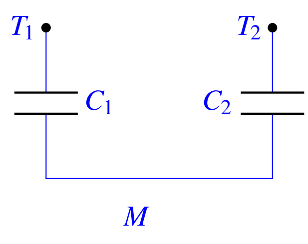
$$C_1 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between T}_1 \text{ and M)}$$

$$C_2 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between T}_2 \text{ and M)}$$

(b) Any circuit on the train can only make direct contact at T_1 and T_2. Thus, you can only measure the equivalent capacitance between T_1 and T_2. Draw a circuit model showing how the capacitors between T_1 and M and between T_2 and M are connected to each other.

Solution:

The capacitors C_1 and C_2 are in series. To realize this, let's consider the train circuit that is in contact with T_1 and T_2. If there is current entering plate T_1, the same current has to exit plate T_2. Thus, the circuit can be modeled as follows:



(c) Using the same parameters as in part (a), provide an expression for the equivalent capacitance between T_1 and T_2.

Solution:

Since the two capacitors are in series, the equivalent capacitance between T_1 and T_2 is given by:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Thus, we get

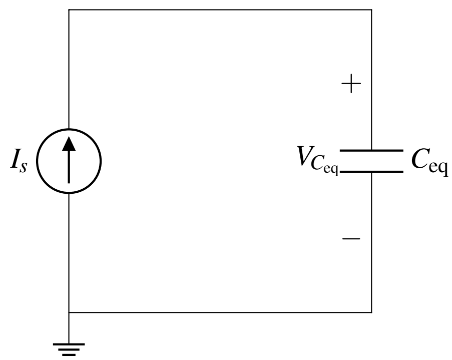
$$\frac{1}{C_{\text{eq}}} = \frac{h}{\varepsilon W L_{\text{train}}} + \frac{h}{\varepsilon W L_{\text{train}}}$$

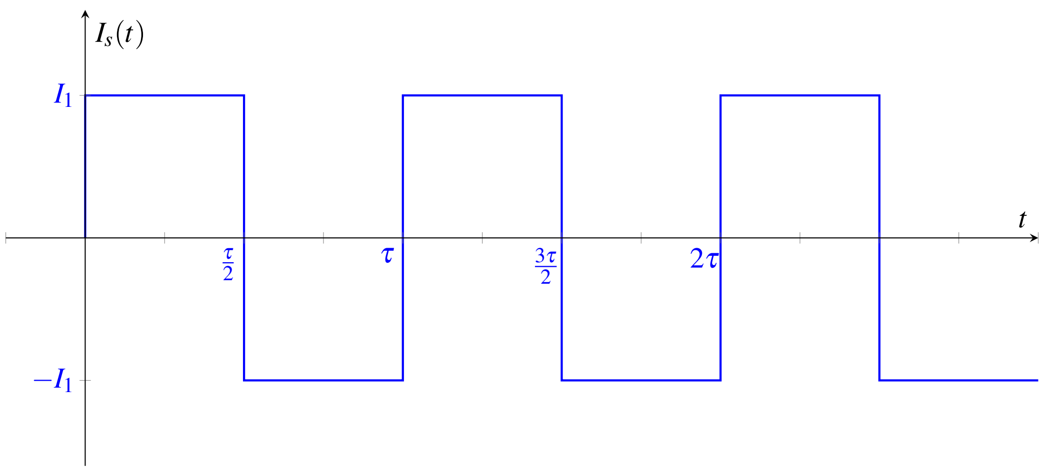
$$C_{\text{eq}} = \frac{\varepsilon W L_{\text{train}}}{2h}$$

(d) We want to build a circuit that creates a voltage waveform with an amplitude that changes based on the height of the train. Your colleague recommends you start with the circuit as shown below, where I_s is a periodic current source, and C_{eq} is the equivalent capacitance between T_1 and T_2. The graph below shows I_s, a square wave with period τ and amplitude I_1, as a function of time.

Find an equation for and draw the voltage $V_{C_{\text{eq}}}(t)$ as a function of time. Assume the capacitor C_{eq} is discharged at time $t = 0$, so $V_{C_{\text{eq}}}(0) = 0\,\text{V}$.

Hint: Your final expression should resemble a periodic function.





Solution: We know the rate of change of voltage across a capacitor is related to the the current into the capacitor. That is:

$$I_{C_{\text{eq}}} = C_{\text{eq}} \frac{dV_{C_{\text{eq}}}}{dt}$$

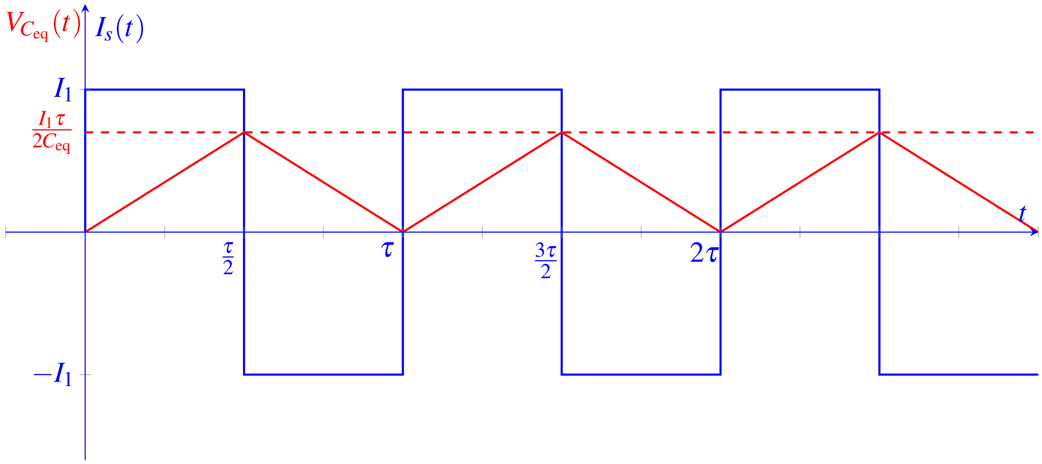
From KCL, we know $I_{C_{\text{eq}}} = I_s$. Then:

$$I_{C_{\text{eq}}} = I_s = C_{\text{eq}} \frac{dV_{C_{\text{eq}}}}{dt} \implies \frac{dV_{C_{\text{eq}}}}{dt} = \frac{I_s}{C_{\text{eq}}}$$

Since I_s is periodic, we can apply the procedure detailed in Note 17, Section 17.2.1 to get the following equation for $V_{C_{eq}}(t)$ for the first period, which repeats for subsequent periods. We recall that the capacitor is uncharged at $t = 0$ so that $V_{C_{eq}}(0) = 0\,\text{V}$.

$$V_{C_{eq}}(t) = \begin{cases} \frac{I_1}{C_{eq}} t & \text{when } 0 \le t \le \frac{\tau}{2} \\ \frac{-I_1}{C_{eq}}\left(t - \frac{\tau}{2}\right) + \frac{I_1 \tau}{2C_{eq}} & \text{when } \frac{\tau}{2} < t \le \tau \end{cases}$$

Given this equation for the output voltage, $V_{C_{eq}}(t)$, as a function of the current, I_s, we can draw what the output waveform should look like.

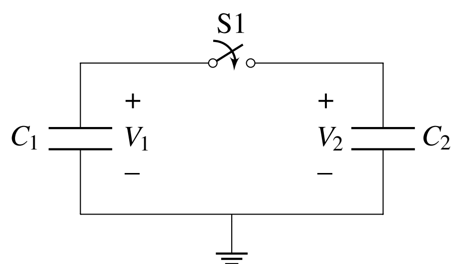


5. **Charge Sharing**

In the circuit below, switch S_1 is initially open. Capacitor $C_1 = 10^{-3}$F is initially charged to $V_1 = 3$V and capacitor $C_2 = 3 \times 10^{-3}$F is initially charged to $V_2 = 2$V.

Now S_1 is closed. Calculate the new value of V_2.

Hint: Remember that charge is conserved at floating nodes.



Solution:

Let us define the initial charge on C_1 as Q_{1i} and the initial charge on C_2 as Q_{2i}. We know that $Q_{1i} = C_1 V_{1i}$ and $Q_{2i} = C_2 V_{2i}$, where V_{1i} and V_{2i} are the initial voltages across C_1 and C_2, respectively. (i.e. before switch S_1 is closed). Notice that once the switch is closed, the two positive plates are connected by a floating node. The positive charges from the two plates in phase 1 will remain on this floating node. This is why we apply charge conservation across the two phases at this node. We know from conservation of charge that

$Q_{1i}+Q_{2i}=Q_{1f}+Q_{2f}$, where Q_{1f} and Q_{2f} are the final charge on C_1 and C_2. (i.e. after switch S_1 is closed). We can write this as:

$$C_1V_{1i}+C_2V_{2i}=Q_{1f}+Q_{2f} \tag{3}$$

$$10^{-3}F\times 3V+3\times 10^{-3}F\times 2V=Q_{1f}+Q_{2f}9\times 10^{-3}C=Q_{1f}+Q_{2f} \tag{4}$$

Additionally, we know that once switch S_1 is closed, the voltage across C_1 and C_2 must be the same, because they are now in parallel with each other. Specifically, $V_{1f}=V_{2f}$ where V_{1f} and V_{2f} are the final voltages across C_1 and C_2, respectively. (i.e. after switch S_1 is closed). We can write this as:

$$V_{1f}=\frac{Q_{1f}}{C_1}=V_{2f}=\frac{Q_{2f}}{C_2} \tag{5}$$

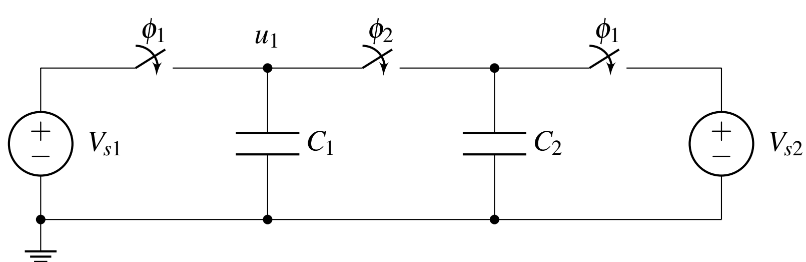
$$V_{1f}=\frac{Q_{1f}}{10^{-3}F}=\frac{Q_{2f}}{3\times 10^{-3}F} \tag{6}$$

We have two equations, and two unknowns (Q_{1f} and Q_{2f}). Using either back substitution, Gaussian elimination, or iPython, we find that $Q_{1f}=0.00225\text{C}$ and $Q_{2f}=0.00675\text{C}$. We can then calculate $V_{2f}=\frac{Q_{2f}}{C_2}=\frac{0.00675\text{C}}{0.003\text{F}}=2.25\text{V}$.

(Also, note that $V_{1f}=\frac{Q_{1f}}{C_1}=\frac{0.00225\text{C}}{0.001\text{F}}=2.25\text{V}$.)

6. Capacitive Charge Sharing

Consider the circuit below with $C_1=C_2=1\,\mu\text{F}$ and two switches ϕ_1,ϕ_2. Suppose that initially the switch ϕ_1 is closed and ϕ_2 is open such that C_1 and C_2 are charged through the corresponding voltage sources $V_{s1}=1\,\text{V}$ and $V_{s2}=2\,\text{V}$.



(a) How much charge is on C_1 and C_2? How much energy is stored in each of the capacitors? What is the total stored energy?

Solution:

$$q_1=C_1V_1=1\,\mu\text{C}$$

$$q_2=C_2V_2=2\,\mu\text{C}$$

Energy:

$$E=\frac{1}{2}CV^2=\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}QV$$

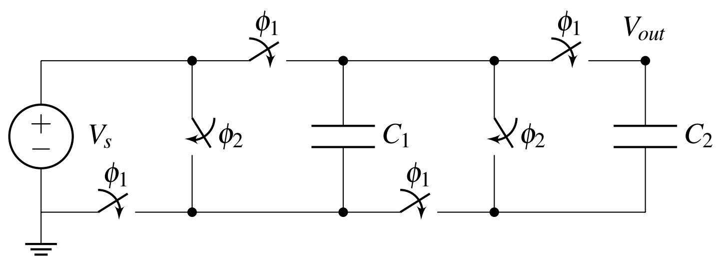
Therefore, $E_1=\frac{1}{2}C_1V_1^2=0.5\,\mu\text{J}$, $E_2=\frac{1}{2}C_2V_2^2=2\,\mu\text{J}$, and the total energy is $2.5\,\mu\text{J}$.

(b) Now suppose that some time later, switch ϕ_1 opens and switch ϕ_2 closes. What is the value of voltage u_1 at steady state?

Solution: The total charge on capacitors C_1 and C_2 will be conserved after switch S_1 is opened. That charge is $Q_tot = q_1 + q_2$. Also note that during phase 2 the capacitors are connected in parallel so they will both have $V_{C1} = V_{C2} = u_1$.

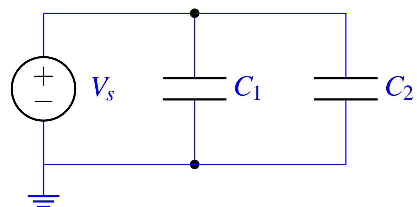
$$Q_{tot} = C_1 u_1 + C_2 u_1$$
$$u_1 = \frac{q_1 + q_2}{C_1 + C_2} = 1.5\,\mathrm{V}$$

(c) Now let's look at the following circuit:

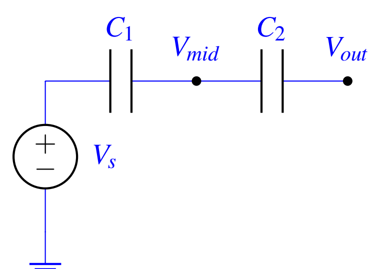


What is the value of voltage V_{out} at the end of phase 2 (steady state)? Does it depend on the values of capacitors C_1 and C_2? *Hint:* It may be useful to redraw the circuit during phases ϕ_1 and ϕ_2.

Solution: Rewriting the circuit in phase 1 we have:



Rewriting the circuit in phase 2 we have:



Notice that in phase 2, there are floating nodes V_{out} and V_{mid}. Thus, the charge will be conserved on nodes V_{out} and V_{mid}.

Identifying all the capacitor plates that are connected on those nodes during phase 1 and calculating the charge onto them in phase 1 we have that:

Node V_{mid}:

$$Q^{\phi_1}_{V_{mid}} = (V_s - 0)C_1 - (V_s - 0)C_2$$

Node V_{out}:

$$Q^{\phi_1}_{V_{out}} = (V_s - 0)C_2$$

Looking at phase 2 and calculating the charge onto the same nodes we have: Node V_{mid}:

$$Q_{V_{mid}}^{\phi_2} = (V_{mid} - V_s)C_1 - (V_{out} - V_{mid})C_2$$

Node V_{out}:

$$Q_{V_{out}}^{\phi_2} = (V_{out} - V_{mid})C_2$$

Equating the charge on the corresponding plates for phases 1 and 2 we get a 2x2 system of linear equations:

$$V_sC_1 - V_sC_2 = (V_{mid} - V_s)C_1 - (V_{out} - V_{mid})C_2 \quad (7)$$
$$V_sC_2 = (V_{out} - V_{mid})C_2 \quad (8)$$

Substituting (2) into (1) yields:

$$V_sC_1 - V_sC_2 = (V_{mid} - V_s)C_1 - V_sC_2$$

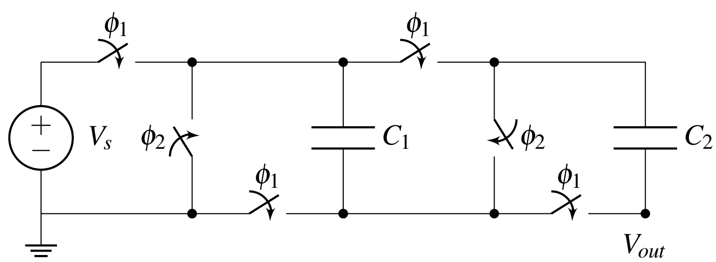
$$V_{mid} = 2V_s$$

Finally substituting V_{mid} into (2):

$$V_{out} = 3V_s$$

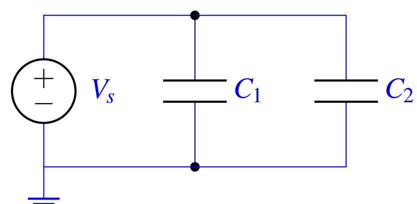
Which means that we managed to create $3V_s$ with a voltage source of only V_s!

(d) A variation of the circuit from part c is shown below:

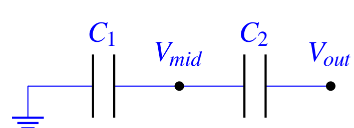


What is the value of voltage V_{out} at the end of phase 2 (steady state)? Does it depend on the values of capacitors C_1 and C_2? *Hint:* It may be useful to redraw the circuit during phases ϕ_1 and ϕ_2.

Solution: Rewriting the circuit in phase 1 we have:



Rewriting the circuit in phase 2 we have:



Notice again that in phase 2, there are floating nodes V_{out}, V_{mid}. So the charge will once again be conserved on nodes V_{out}, V_{mid}, but now the polarities of the capacitor (i.e. which plate is positive and which side is negative) are different (due to how we configured the switches).

Identifying all the capacitor plates that are connected on those nodes during phase 1 and calculating the charge onto them in phase 1 we have that:

Node V_{mid}:

$$Q^{\phi_1}_{V_{mid}} = -(V_s - 0)C_1 + (V_s - 0)C_2$$

Node V_{out}:

$$Q^{\phi_1}_{V_{out}} = -(V_s - 0)C_2$$

Looking at phase 2 and calculating the charge onto the same nodes we have: Node V_{mid}:

$$Q^{\phi_2}_{V_{mid}} = -(0 - V_{mid})C_1 + (V_{mid} - V_{out})C_2$$

Node V_{out}:

$$Q^{\phi_2}_{V_{out}} = -(V_{mid} - V_{out})C_2$$

Equating the charge on the corresponding plates for phases 1 and 2 we get a 2x2 system of linear equations:

$$-V_sC_1 + V_sC_2 = -V_{mid}C_1 + (V_{mid} - V_{out})C_2 \quad (9)$$

$$-V_sC_2 = -(V_{mid} - V_{out})C_2 \quad (10)$$

Substituting (4) into (3) yields:

$$-V_sC_1 + V_sC_2 = V_{mid}C_1 + V_sC_2$$

$$V_{mid} = -V_s$$

Finally substituting V_{mid} into (4):

$$V_{out} = -2V_s$$

In this case we have created a negative voltage of $-2V_s$ using just a voltage source of V_s!

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.